MAU44406 - The Standard Model of Elementary Particle Physics
Brief Notes

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The Standard Model is so complex it would be hard to put it on a T-shirt — though not impossible; you'd just have to write kind of small.

Steven Weinberg

1 Introduction

There is material covered in the course that is not covered here, mostly because they are asides to the main content, don't fit well into the structure of these notes, or are easily looked up from your own notes or on Wikipedia.

$\hbar = c = 1$.

1.1 What is the Standard Model?

The Standard Model of Elementary Particle Physics is the culmination of over 100 years of particle physics research and theory. It describes the interactions of all known matter through the fundamendal forces of electromagnetism, the weak nuclear force, and the strong nuclear force but *not* gravity. Why is this the case? In straightforward terms, it's because gravity couples incredibly weakly to everyday particles like electrons, quarks, protons, etc. A back-of-the-envelope calculation of a hydrogen atom will tell you that a proton and electron separated by a distance r feels a Coulomb force

$$F_C = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \tag{1.1}$$

and a gravitational force

$$F_G = \frac{Gm_e m_p}{r^2}. \tag{1.2}$$

The ratio of these two forces is

$$\frac{F_G}{F_C} \sim 10^{-40}. \tag{1.3}$$

The effects of gravity are wholly irrelevant when dealing with subatomic particles.

1.2 The Standard Model Lagrangian

The Lagrangian density of the Standard Model is composed of interaction terms for each of the major particle groups

$$\mathcal{L}_{\text{SM}}(x) = \mathcal{L}_{\text{quarks}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} \tag{1.4}$$

The quark term couples the quarks to the $W^{\pm}$ and Z bosons, gluons, and photons.

$$\mathcal{L}_{\text{quarks}}(x) = \sum_{i=1}^{3} \bar{\psi}_{q,i}(x) \not{D} \psi_{q,i}(x), \quad \psi_{q,i} = \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}, \quad \not{D} = \gamma_\mu D^\mu \ni W^{\pm}, Z, g, \gamma. \tag{1.5}$$

The lepton term couples the leptons to the $W^{\pm}$ and Z bosons, and photons but not to the gluons.

$$\mathcal{L}_{\text{leptons}}(x) = \sum_{i=1}^{3} \bar{\psi}_{l,i}(x) P_L \not{D} \psi_{l,i}(x), \quad \psi_{l,i} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}. \tag{1.6}$$

The Higgs term consists of a doublet of complex scalar fields describing the Higgs and the Goldstone bosons. It couples to the $W^{\pm}$ and Z bosons and photons, but not gluons.

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger D^\mu \phi - V(\phi), \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{1.7}$$

The Yukawa term couple the matter fields (quarks and leptons) to the Higgs field. There is no neutrino term as they are approximated as massless particles.*

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} = & \sum_{i,j=1}^{3} y^l_{ij} \bar{\psi}_{l,i} P_L \phi P_R \begin{pmatrix} e^- \\ \mu^- \\ \tau^- \end{pmatrix}_j + \text{Hermitian conjugate} \\ & + \sum_{i,j=1}^{3} y^u_{ij} \bar{\psi}_{q,i} P_L \widetilde{\phi} P_R \begin{pmatrix} u \\ c \\ t \end{pmatrix}_j + \text{h. c.} \\ & + \sum_{i,j=1}^{3} y^d_{ij} \bar{\psi}_{q,i} P_L \phi P_R \begin{pmatrix} d \\ s \\ b \end{pmatrix}_j + \text{h. c.} \end{aligned} \tag{1.8}$$

2 The Klein-Gordon and Dirac Equations

2.1 The Klein-Gordon Equation

The Schrödinger equation is based on the non-relativistic energy-momentum relation

$$E = \frac{\mathbf{p}^2}{2m} \quad \text{with} \quad E \to i\frac{\partial}{\partial t}, \quad \mathbf{p} \to -i\nabla. \tag{2.1}$$

A relativistic wave equation can be constructed from the relativistic energy-momentum relation

$$p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2. \tag{2.2}$$

Using the same identifications as above, we get

$$-\frac{\partial^2}{\partial t^2}\phi(\mathbf{x}, t) = (-\nabla^2 + m^2)\phi(\mathbf{x}, t). \tag{2.3}$$

*As you may know, neutrinos do indeed have nonzero mass although they are incredibly light. Most estimates put their mass at less than 1 eV. Many experiments are currently ongoing to better measure the neutrino mass and to determine more of their properties, e.g. the KATRIN experiment.

Recognising the d'Alembertian $\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$, we get the Klein-Gordon equation

$$(\Box + m^2)\phi(\mathbf{x}, t) = 0. \tag{2.4}$$

The solutions to this equation are plane waves

$$\phi(\mathbf{x}, t) = Ne^{-ip\cdot x} = Ne^{-iEt+i\mathbf{p}\cdot\mathbf{x}} \tag{2.5}$$

Note that for a given mass and momentum $\mathbf{p}, m$, there are two energy solutions $E = \pm\sqrt{\mathbf{p}^2 + m^2}$.

The probability current

$$(\rho, \mathbf{j}) = j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \tag{2.6}$$

satisfies the continuity equation $\partial_\mu j^\mu = 0$. In Quantum Mechanics, the time derivative of ρ would be interpreted as a probability density. For the Klein-Gordon plane waves, we find

$$\rho = i|N|^2(e^{ip\cdot x}(-iE)e^{-ip\cdot x} - e^{-ip\cdot x}(iE)e^{ip\cdot x}) = 2|N|^2 E \tag{2.7}$$

however, E can take negative values. These "negative probability" solutions prevent the usual quantum mechanical interpretation of a probability current.

When this equation was initially published in 1926, Klein and Gordon asserted that it could describe relativistic electrons however this is not the case as modelling the electron requires the equation to take its spin into account. The equation does however model spinless particles such as the Higgs boson, although this only came about much later in the 20th Century.

As a result, the Klein-Gordon equation was abandoned at first, until it was later used in Quantum Field Theory where the negative solutions are considered charge densities rather than probability densities.

2.2 The Dirac Equation

Dirac proposed that the problems associated with the Klein-Gordon equation could be due to the fact that it has a second order derivative in time, unlike the Schrödinger equation which is first order. He proposed the ansatz

$$i\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = (-i\alpha \cdot \nabla + \beta m)\psi(\mathbf{x}, t) \tag{2.8}$$

for some undetermined coefficients $\alpha_1, \alpha_2, \alpha_3, \beta$. These coefficients must satisfy the relativistic energy-momentum relation and ensure that the equation is Lorentz invariant. To satisfy the first of these, Dirac demanded that ψ satisfy the Klein-Gordon equation. This requires

$$\begin{gathered} \{\alpha_i, \alpha_j\} = 0, \quad (i \neq j) \\ \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1 \\ [\alpha_i, \beta] = 0. \end{gathered} \tag{2.9}$$

Dirac soon figured that scalars wouldn't work, and the Pauli matrices almost do, but they don't anticommute with β. Instead, these coefficients must be the 4×4 gamma matrices

$$\begin{gathered} \beta = \gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha^1 = \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ \alpha^2 = \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \alpha^3 = \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{gathered} \tag{2.10}$$

These 4×4 matrices require that ψ is a four-component complex-valued vector called a spinor:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{2.11}$$

The probability current for the Dirac equation can be constructed in a similar way to the Schrödinger equation by multiplying to the left by $\psi^\dagger$. This gives the current and charge densities

$$\rho = \psi^\dagger \psi, \quad j^k = \psi^\dagger \alpha^k \psi \tag{2.12}$$

which satisfy the continuity equation $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$. This equation implies that probability is conserved:

$$\frac{\partial}{\partial t} \int d^3x \rho(\mathbf{x}, t) = 0. \tag{2.13}$$

In the non-relativistic limit for a stationary electron, the Dirac equation reproduces the Pauli equation for a spin-$\frac{1}{2}$ particle with the correct magnetic moment; a major achievement for the theory.

The Dirac equation can be put into a covariant form using "slashed" notation $\not{a} = \gamma_\mu a^\mu$

$$(i\not{\partial} - m)\psi = 0. \tag{2.14}$$

The adjoint form can also be written using $\bar{\psi} = \psi^\dagger \gamma^0$

$$\bar{\psi}(-i\not{\partial} - m) = 0. \tag{2.15}$$

3 Gauge Symmetries

3.1 Gauge Invariance and QED

Consider the Lagrangian of a Dirac spinor (an electron) and the Maxwell electromagnetic field strength tensor

$$\mathcal{L}(x) = \bar{\psi}(i\not{\partial} - m + e\not{A}(x))\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3.1}$$

This Lagrangian is invariant under the combined transformations

$$\begin{aligned} \psi(x) &\to e^{i\omega(x)} \psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}(x) e^{-i\omega(x)} \\ A_\mu(x) &\to A_\mu(x) + \frac{1}{e} \partial_\mu \omega(x). \end{aligned} \tag{3.2}$$

Defining the gauge covariant derivative[†]

$$D_\mu = \partial_\mu - ieA_\mu, \tag{3.3}$$

we can write the Dirac term in the Lagrangian as

$$\mathcal{L}(x) = \bar{\psi}(x)(i\not{D} - m)\psi(x) \tag{3.4}$$

[†] This concept of a covariant derivative is almost identical to that of the covariant derivative in General Relativity. The difference here is that instead of the Christoffel connection $\Gamma^\rho_{\mu\nu}$, we have the electromagnetic potential A_μ. There are no extra indices on this connection because the gauge symmetry for EM is the one-dimensional space $U(1)$ rather than the Lorentz group $O(3,1)$. Additionally, A_μ is an affine connection which does not require a metric tensor, since we cannot define a metric for gauge theories in particle physics. Particle physics gauge groups are "added on" after the fact (as an internal fiber bundle), while the Christoffel connection is directly related to the tangent bundles of the spacetime manifold itself.

We have $D_\mu\phi(x) \to D'_\mu\phi'(x) = e^{i\omega(x)}D_\mu\phi(x)$ under this transformation, so $\bar{\psi}D_\mu\psi$ in the Lagrangian is gauge invariant.

We can invert the above procedure by considering a global symmetry where ω is coordinate independent (i.e. global).

$$\begin{aligned} \psi(x) &\to e^{i\omega}\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{-i\omega}. \end{aligned} \tag{3.5}$$

The Lagrangian is invariant under this transformation as $e^{i\omega}$ is now just some number. We can promote the global symmetry to a local one by making it coordinate dependent: $\omega = \omega(x)$. To ensure the Lagrangian is invariant we introduce a gauge field

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu \tag{3.6}$$

where q is a coupling constant ($q = -e$ for electrons). We can find the transformation rule from the constraint that this covariant derivative must leave the Lagrangian invariant by construction. We find that we recover exactly the same transformation rule for A_μ as above, namely

$$A_\mu \to A_\mu - \frac{1}{q}\partial_\mu\omega(x). \tag{3.7}$$

We now see that we can make the global $U(1)$ symmetry of the Lagrangian a local symmetry by introducing the gauge field $A_\mu(x)$ which couples to the fermion. This scheme also works for scalar fields.

The inclusion of a gauge field in the Lagrangian is essentially a self-interaction term. We can write

$$\begin{aligned} \mathcal{L}(x) &= \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(i(\gamma^\mu\partial_\mu + iq\gamma^\mu A_\mu) - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(i\not{\partial} - m)\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{3.8}$$

The additional middle term is the interaction of the fermions with the electromagnetic field. We can see this more clearly using the Euler-Lagrange equations.

$$0 = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\mu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - q\bar{\psi}\gamma^\mu\psi. \tag{3.9}$$

Imposing the Lorenz gauge condition $\partial_\nu A^\nu = 0$, we have

$$\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\partial_\nu\partial^\nu A^\mu = -\Box A^\mu. \tag{3.10}$$

Thus we have the equation $\Box A^\mu = -q\bar{\psi}\gamma^\mu\psi$. Recognising this as the Dirac current $j^\mu = -q\bar{\psi}\gamma^\mu\psi$, as in (2.12), we have

$$\Box A^\mu = j^\mu \tag{3.11}$$

which is a wave equation with a charge term. This is a version of the Maxwell equations for minimal coupling in Quantum Electrodynamics (QED). The introduction of a gauge field naturally makes interactions manifest.

3.2 Non-Abelian Gauge Symmetries

The gauge group for electromagnetism, $U(1)$ is an Abelian (commutative) group, however there are Lagrangians which have non-abelian symmetries. We can perform the same procedure. Take for example a Dirac spinor with two components representing a proton and neutron

$$\psi = \begin{pmatrix} \psi_n \\ \psi_p \end{pmatrix}. \tag{3.12}$$

The free Dirac Lagrangian (in matrix form) is

$$\mathcal{L}(x) = (\bar{\psi}_n \; \bar{\psi}_p) \begin{bmatrix} i\partial\!\!\!/ - m_n & 0 \\ 0 & i\partial\!\!\!/ - m_p \end{bmatrix} \begin{pmatrix} \psi_n \\ \psi_p \end{pmatrix}. \tag{3.13}$$

If we assume $m_n = m_p = m$ then the Lagrangian has a global $SU(2)$ invariance. That is, for a 2×2 matrix $U \in SU(2)$ satisfying $U^\dagger U = \mathbb{1}$ and $\det U = 1$, we have

$$\mathcal{L}(x) \to (\bar{\psi}_n \; \bar{\psi}_p) \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^\dagger \begin{bmatrix} i\partial\!\!\!/ - m_n & 0 \\ 0 & i\partial\!\!\!/ - m_p \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{pmatrix} \psi_n \\ \psi_p \end{pmatrix}. \tag{3.14}$$

If we apply the gauge principle to this Lagrangian and promote U to a local symmetry $U(x)$, then to preserve the $SU(2)$ symmetry we require the gauge covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu \tag{3.15}$$

for a gauge field A_μ with coupling constant g. We can expand the gauge field in the Pauli matrices τ^a, which form a basis of $SU(2)$.

$$A_\mu(x) = \sum_{a=1}^{3} A_\mu^a(x) \frac{\tau^a}{2} \tag{3.16}$$

The components A_μ^a are real-valued vector fields.

We can recover the field strength tensor from its definition in terms of the covariant derivative

$$\begin{aligned} [D_\mu, D_\nu]\psi &= [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]\psi \\ &= \left\{[\partial_\mu, \partial_\nu] + ig(\partial_\mu A_\nu - \partial_\nu A_\mu) + igA_\mu\partial_\nu - igA_\nu\partial_\mu + igA_\nu\partial_\mu - igA_\mu\partial_\nu - g^2[A_\mu, A_\nu]\right\}\psi \\ &= ig\left\{\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\right\}\psi \\ &:= igF_{\mu\nu}\psi. \end{aligned} \tag{3.17}$$

We have the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \tag{3.18}$$

In the Abelian case where A_μ and A_ν commute, we recover the usual Maxwell field strength tensor.

4 Quantum Chromodynamics

4.1 Yang-Mills Theory

The field strength tensor in the non-Abelian case is not gauge invariant or Lorentz invariant. It transforms as

$$F_{\mu\nu}(x) \to U(x)F_{\mu\nu}(x)U^\dagger(x). \tag{4.1}$$

What we want for gauge invariance is a Lagrangian which is, well, gauge invariant. We also want it to be Lorentz invariant so it is compatible with our usual notions of special relativity. We can be satisfied that it is invariant up to total derivatives, since they will not contribute to the physical action.

We could consider trying the quantity

$$F_{\mu\nu}F^{\mu\nu} \to UF_{\mu\nu}U^\dagger UF^{\mu\nu}U^\dagger = UF_{\mu\nu}F^{\mu\nu}U^\dagger. \tag{4.2}$$

This quantity is Lorentz invariant, but not gauge invariant. We can ensure gauge invariance by taking the trace

$$\mathcal{L}(x) \sim \frac{1}{2}\,\mathrm{tr}\left(F_{\mu\nu}(x)F^{\mu\nu}(x)\right). \tag{4.3}$$

This quantity appears in the Yang-Mills Lagrangian, a generalisation of Maxwell theory, which describes the action of elementary particles coupled to an $SU(2)$ gauge field.

$$\mathcal{L}(x) = -\frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\not{D} - m)\psi + (D_\mu\phi)^\dagger D^\mu\phi + m^2\phi^\dagger\phi \tag{4.4}$$

The first term is the gauge invariant field strength tensor as before, the second term couples the fermions, and the third term couples bosons. ψ and ϕ are both doublets under $SU(2)$ transformations.

A general Yang-Mills theory applies this idea to the $SU(N)$ group.

4.2 Quarks and QCD

A single quark $\psi_q(x)$ comes in three "colours"; red, green, and blue and its spinor is

$$\psi_q(x) = \begin{pmatrix} \psi_q^{\text{blue}}(x) \\ \psi_q^{\text{red}}(x) \\ \psi_q^{\text{green}}(x) \end{pmatrix} = \psi_q^i(x). \tag{4.5}$$

Gauge symmetry transformations are in the $SU(3)$ group acting in colour space. We can use the Yang-Mills Lagrangian.

$$\mathcal{L}(x) = -\frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}_q(i\not{D} - m_q)\psi. \tag{4.6}$$

In nature there are six quarks, each with a different flavour: up, down, strange, charm, top, and bottom. The colour theory of these quarks is described by Quantum Chromodynamics. The QCD Lagrangian is

$$\mathcal{L}_{\text{QCD}}(x) = \sum_{f=u,d,s,c,t,b} \bar{\psi}_f(i\not{D} - m_f)\psi_f - \frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{4.7}$$

Here the covariant derivative is

$$D_\mu = \partial_\mu + igA_\mu(x), \quad A_\mu(x) = \sum_{a=1}^{8} A_\mu^a(x)T^a = \sum_{a=1}^{8} A_\mu^a(x)\frac{\lambda^a}{2}, \tag{4.8}$$

where λ^a are the Gell-Mann matrices forming a basis of $SU(3)$. QCD describes the strong force and its mediation by gluons.

4.3 QCD + QED

We can combine QCD with QED by considering the electric charges of quarks.

$$\begin{aligned} q_u &= q_c = q_t = \tfrac{2}{3}e \\ q_d &= q_s = q_b = -\tfrac{1}{3}e. \end{aligned} \tag{4.9}$$

To include the extra electromagnetism gauge symmetry, we have to expand the covariant derivative to couple with the photon.

$$D_\mu = \partial_\mu + igA_\mu^a T^a + ig_p A_\mu \tag{4.10}$$

Here, A_μ^a is the $SU(3)$ matrix gauge field and A_μ is the photon $U(1)$ gauge field. The combination of QCD and QED gives the Lagrangian

$$\mathcal{L}_{\text{QCD+QED}} = \sum_{\alpha,\beta}\sum_{i,j}\sum_{f,f'}\bar{\psi}_{\alpha,f,i}\{i\gamma^\mu_{\alpha\beta}(\delta_{ij}\partial_\mu + ig\sum_{a=1}^{8}A_\mu^a T_{ij}^a + iQ_{ff'}A_\mu\delta_{ij})\delta_{ff'} - M_{ff'}\delta_{ij}\delta_{\alpha\beta}\}\psi_{\beta,f',j} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\sum_{a=1}^{8}F_{\mu\nu}^a F^{\mu\nu,a}. \tag{4.11}$$

Here, α and β run over the spinor indices, i and j run over the three colour indices red, blue, green, and f and f' run over the flavour indices. We have also defined the quark spinor

$$\psi_\alpha = (\psi_u \;\; \psi_d \;\; \psi_c \;\; \psi_s \;\; \psi_t \;\; \psi_b)^T, \tag{4.12}$$

the mass matrix $M = \text{diag}(m_u, m_d, m_c, m_s, m_t, m_b)$ and the charge matrix $Q = \text{diag}(q_u, q_d, q_c, q_s, q_t, q_b)$.

A_μ^a are the gluons and A_μ is the photon. $F_{\mu\nu}$ is the stress tensor for photons and $F_{\mu\mu}^a$ is for the gluons.

The QCD Lagrangian has several symmetries. It is $SU(3)$ invariant by construction. There is also a flavour symmetry; the gluons are "flavour blind" since the coupling constant g is the same for all of them.

If we consider quarks whose masses are all equal, then $M = m\mathbb{1}_6$ and there is a global $U(6)$ symmetry. In reality, the masses are not equal. There are some approximations that can be made such as $m_u \simeq m_d \sim 1$ MeV, but the other quark masses are too different to give reasonable predictions this way: $m_s \sim 100$ MeV, $m_c \sim 1$ GeV, $m_b \sim 5$ GeV, $m_t \sim 176$ GeV.

The symmetry between the up and down quarks is referred to as isospin, a name derived from an older attempt to quantify the angular momenta of the proton and neutron.

4.4 CP Symmetry in QCD

If we consider a Lagrangian

$$\mathcal{L} = \bar{\psi}i\not{D}\psi, \tag{4.13}$$

we can introduce projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$, where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, which satisfy $P_+ + P_- = \mathbb{1}$. Thus, we can write

$$\mathcal{L} = \bar{\psi}(P_+ + P_-)i\not{D}(P_+ + P_-)\psi = \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R \tag{4.14}$$

where $\psi_L = P_-\psi, \bar{\psi}_L = \bar{\psi}P_+$ and $\psi_R = P_+\psi, \bar{\psi}_R = \bar{\psi}P_-$ are projected fields. We can perform independent $SU(2)$ transformations on the left and right handed fields. This is the $SU(2)_L \times SU(2)_R$ symmetry.

Mass terms, however, break chiral symmetry. Take for example

$$m\bar{\psi}\psi = m\bar{\psi}(P_+ + P_-)\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L). \tag{4.15}$$

Performing separate transformations yields

$$\bar{\psi}_L\psi_R \to \bar{\psi}_L U_L^\dagger U_R\psi_R, \quad \bar{\psi}_R\psi_L \to \bar{\psi}_R U_R^\dagger U_L\psi_L. \tag{4.16}$$

The mass acts as a coupling term between left and right handed fields and $SU(2)_L \times SU(2)_R$ is explicitly broken unless $U_L = U_R$, which is $SU(2)$ isospin.

The QCD Lagrangian is not symmetric under parity transformations

$$P : x^\mu \to \widetilde{x}^\mu = (x^0, -\mathbf{x}) = (-1)^{1+\delta_{\mu 0}} x^\mu \tag{4.17}$$

however the action itself is invariant, so QCD is said to have parity symmetry *unless* one includes a stress-energy term

$$\Theta \frac{1}{2} \operatorname{tr}(\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} F^{\mu\nu}) \tag{4.18}$$

In Maxwell theory, this term is $\mathbf{E} \cdot \mathbf{B}$, which changes sign under parity. If there is such a term in the strong interaction, it must be extremely small as no CP violations have been observed in strong decays. This is known as the strong CP problem and is related to a more general issue of "fine-tuning" where certain model parameters must be very carefully tuned in order to fit experiment. The Standard Model is an excellent predictive theory, but does not offer explanations as to *how* different phenomena arise or why particles have the masses they do.

5 The Weak Nuclear Force

5.1 Weak Interactions

The weak interaction is based on an experimentally observed local $SU(2)_L$ symmetry. The fermions are grouped into left- and right-handed fields

$$\begin{gathered} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_{L,R}, \quad \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix}_{L,R}, \quad \begin{pmatrix} \psi_t \\ \psi_b \end{pmatrix}_{L,R}, \\ \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_{L,R}, \quad \begin{pmatrix} \psi_{\nu_\mu} \\ \psi_\mu \end{pmatrix}_{L,R}, \quad \begin{pmatrix} \psi_{\nu_\tau} \\ \psi_\tau \end{pmatrix}_{L,R}. \end{gathered} \tag{5.1}$$

The left-handed fields are doublets under $SU(2)_L$ transformations

$$\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L \to U_L \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L \tag{5.2}$$

and the right-handed fields are singlets (they do not transform under $SU(2)_L$).

There are some problems that arise here. If $SU(2)_L$ is to become a local symmetry, then it must be exact, however mass terms break the symmetry. We also need to find a way to incorporate electromagnetism for the charged fermions.

5.2 The Glashow-Weinberg-Salam Model

We consider the leptons of the first family (it's the same for the others). Approximating neutrinos as massless, the free Lagrangian density is

$$\mathcal{L}(x) = (\bar{\psi}_{\nu_e} \quad \psi_e)_L i\partial\!\!\!/ \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L + \bar{\psi}_{e,R} i\partial\!\!\!/ \psi_{e,R} \tag{5.3}$$

We consider a $SU(2)_L$ transformation and promote it to a local symmetry with the introduction of a $SU(2)_L$ gauge field

$$W_\mu(x) = \sum_{a=1}^{3} W_\mu^a(x) \frac{\tau^a}{2}. \tag{5.4}$$

The field strength is

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu]. \tag{5.5}$$

The Lagrangian is then

$$\mathcal{L}(x) = -\frac{1}{2}\,\mathrm{tr}(W_{\mu\nu}(x)W^{\mu\nu}(x)) + (\bar{\psi}_{\nu_e}\ \bar{\psi}_e)_L\gamma^\mu(\partial_\mu + igW_\mu)\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix} + \bar{\psi}_{e,R}i\partial\!\!\!/\psi_{e,R}. \tag{5.6}$$

We can incorporate electromagnetism through the addition of a global hypercharge symmetry $U(1)_Y \subseteq U(1)_L \times U(1)_R$ where

$$\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}_L \to e^{iY_L\chi}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}_L, \quad \psi_{e,R} \to e^{iY_R\chi}\psi_{e,R}. \tag{5.7}$$

$Y_{L,R}$ are the hypercharges. We can promote this global symmetry to a local one with $\xi \to \xi(x)$ and introducing an abelian gauge field B_μ. The Lagrangian then becomes

$$\mathcal{L}(x) = -\frac{1}{2}\,\mathrm{tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \bar{\psi}iD\!\!\!\!/\,\psi \tag{5.8}$$

where $\psi(x) = (\psi_{\nu_e,L}\ \ \psi_{e,L}\ \ \psi_{e,R})$ and the covariant derivative is

$$D_\mu = \partial_\mu + ig\sum_{a=1}^{3}W_\mu^a T^a + ig' B_\mu Y \tag{5.9}$$

here $Y = \mathrm{diag}(Y_L, Y_L, Y_R)$.

The terms in W_μ^3 and B_μ will give the Z_μ gauge boson and A_μ photon fields.

$$\begin{aligned} Z_\mu &= \frac{1}{\sqrt{g^2+g'^2}}(gW_\mu^3 - g'B_\mu) \\ A_\mu &= \frac{1}{\sqrt{g^2+g'^2}}(g'W_\mu^3 + gB_\mu) \end{aligned} \tag{5.10}$$

Defining the weak mixing angle

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \quad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \tag{5.11}$$

we can write the fermion-gauge boson interaction term as

$$\mathcal{L}_{\mathrm{int}} = -e\left[A_\mu j^\mu_{\mathrm{EM}} + \frac{1}{\sqrt{2}\sin\theta_W}(W_\mu^+\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{e,L} + W_\mu^-\bar{\psi}_{e,L}\gamma^\mu\psi_{\nu_e,L}) + \frac{1}{\sin\theta_W\cos\theta_W}Z_\mu j^\mu_{\mathrm{NC}}\right] \tag{5.12}$$

where the electromagnetic current is

$$j^\mu_{\mathrm{EM}} = -\bar{\psi}_{e,L}\gamma^2\psi_{e,L} - \bar{\psi}_{e,R}\gamma^\mu\psi_{e,R} = -\bar{\psi}_e\gamma^\mu\psi_e \tag{5.13}$$

and the neutral current is

$$j^\mu_{\mathrm{NC}} = \frac{1}{2}\bar{\psi}_{\nu_e,L}\gamma^\mu\psi_{\nu_e,L} - \frac{1}{2}\psi_{e,L}\gamma^\mu\psi_{e,L} - \sin^2\theta_W j^\mu_{\mathrm{EM}}. \tag{5.14}$$

A problem arises here regarding how we can introduce mass terms for the $W^\pm$ and Z bosons without one for the photon, and how to avoid these mass terms breaking the $SU(2)_L \times U(1)_Y$ symmetry of the Lagrangian. This is accomplished by spontaneous symmetry breaking and the Goldstone theorem.

6 The Higgs Sector

6.1 The Goldstone Theorem

Consider a real N-dimensional scalar field

$$\varphi = (\varphi_1 \ \cdots \ \varphi_N)^T \tag{6.1}$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi^T \partial^\mu \varphi - V(\varphi) \tag{6.2}$$

For a set of degenerate vacuum states, we label one as $\varphi = v$. If we consider fluctuations around this minimum, we find

$$\left.\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right|_{\varphi=v} = M_{ij}^2, \tag{6.3}$$

where the eigenvalues of M^2 are the squared masses of physical particle excitations around the vacuum state. It can be shown that some of these eigenvalues are zero, corresponding to massless particle excitation around $\varphi = v$, called Goldstone bosons.

This theorem can be applied to the Higgs Lagrangian.

6.2 The Higgs Lagrangian

The Higgs Lagrangian consists of a complex scalar doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_3 + i\varphi_1 \\ \varphi_4 + i\varphi_2 \end{pmatrix} \tag{6.4}$$

This can be rearranged into an $O(4)$ vector

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}. \tag{6.5}$$

The Lagrangian is

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial_\mu \varphi^T \partial^\mu \varphi - V(\varphi). \tag{6.6}$$

We can write this in terms of a 2×2 matrix $\Sigma = \sqrt{2}(i\tau^2\phi^*\phi)$ to get an equivalent expression

$$\mathcal{L} = \frac{1}{4}\operatorname{tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) - \left[\frac{\mu^2}{4}\operatorname{tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\lambda}{16}\operatorname{tr}\left(\Sigma^\dagger\Sigma\right)^2\right] \tag{6.7}$$

This is manifestly invariant under the $SU(2)_L \times SU(2)_R$ symmetry $\Sigma \to U_L \Sigma U_R^\dagger$.

The number of Golstone bosons is related to the number of broken symmetries of the Lagrangian. The full Lagrangian has $O(4)$ symmetry, with 6 generators. Choosing the vacuum state

$$v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}, \tag{6.8}$$

we see that a rotation $O(3)$ in the first three coordinates leaves the minimum unchanged. There are three generators of $O(3)$, so we have $6 - 3 = 3$ Goldstone bosons. In the original ϕ field, the minimum is at

$$\phi = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{6.9}$$

By considering a perturbation around the minimum of the potential, one can find the mass of a massive scalar field

$$m_H^2 = 2\lambda v^2 \tag{6.10}$$

and three more massless Goldstone bosons.

6.3 Gauging Out the Goldstone Bosons

By decomposing $\Sigma = (v + H)U$ for a real scalar field $v + H$ and $U \in SU(2)$, the Higgs Lagrangian can be written as

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{(v+H)^2}{4} \operatorname{tr}\left[(D_\mu U)^\dagger D^\mu U\right] - V(H), \tag{6.11}$$

where the potential is $V(H) = \frac{1}{2}m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4$. The first term is a mass term and the other two are self-coupling terms of the Higgs field to itself.

The covariant derivative has coupling terms to the $W^\pm$ and Z bosons and to the photon

$$D_\mu = \partial_\mu + igW_\mu + \frac{1}{2}ig' B_\mu. \tag{6.12}$$

By parametrising $U(x) = \exp\Big(i\sum_{a=1}^3 \chi^a(x)\tau^a/v\Big)$, the trace term in the Higgs Lagrangian can be written as

$$\operatorname{tr}\left[(D_\mu U)^\dagger D^\mu U\right] = \frac{1}{2}g^2(W'^1_\mu W'^{\mu,1} + W'^2_\mu W'^{\mu,2}) + \frac{1}{2}W'^3_\mu B_\mu \begin{bmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{bmatrix} \begin{pmatrix} W'^{\mu,3} \\ B^\mu \end{pmatrix} \tag{6.13}$$

This can be diagonalised by the same orthogonal transformation as the GWS model to get

$$\frac{v^2}{4} \operatorname{tr}\left[(D_\mu U)^\dagger D^\mu U\right] = m_W^2 W'^+_\mu W'^{\mu -} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \tag{6.14}$$

where $m_W^2 = \frac{g^2 v^2}{4}$, $m_Z^2 = \frac{g^2 + g'^2}{4}v^2$ and $m_\gamma^2 = 0$. The Goldstone bosons have disappeared with the gauge fixing. We have the final Higgs Lagrangian in the unitary gauge (with the Goldstone bosons “gauged out”) coupled to the $W^\pm$ and Z bosons.

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}\partial_\mu H \partial^\mu H + \left(1 + \frac{H}{v}\right)^2 \left[m_W^2 W'^+_\mu W'^{\mu -} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu\right] - \frac{1}{2}m_H^2 H^2 - \lambda v H^3 - \frac{\lambda}{4}H^4. \tag{6.15}$$

This Lagrangian describes the Higgs boson H, its self-interactions, and its Weak interactions with the $W^\pm$ and Z bosons.

7 Yukawa Theory

7.1 Spontaneous Symmetry Breaking

The Yukawa sector of the Standard Model allows for fermion mass terms to be added to a Lagrangian without explicitly breaking the $SU(2) \times U(1)_Y$ gauge symmetry. These mass terms are introduced by coupling the Higgs doublet to the fermions in the Yukawa Lagrangian. For the electronic fermions, we have

$$\mathcal{L}_{\text{Yukawa}} \sim -c_e \bar{\psi}_{e,R}\phi^\dagger \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L - c_e(\bar{\psi}_{\nu_e}\ \bar{\psi}_e)_L \phi \psi_{e,R} \tag{7.1}$$

The Higgs coupling allows for this Lagrangian to be generally invariant under the $SU(2)_L \times U(1)_Y$ gauge symmetry but in the ground state of the Higgs field, this symmetry is spontaneously broken and allows for mass terms. Specifically, the Lagrangian contains the term

$$-c_e(\tfrac{v}{\sqrt{2}}\bar{\psi}_{e_R}\psi_{e,L} + \tfrac{v}{\sqrt{2}}\bar{\psi}_{e,L}\psi_{e,R}) = -c_e\tfrac{v}{\sqrt{2}}\bar{\psi}_e\psi_e := -m_e\bar{\psi}_e\psi_e \tag{7.2}$$

which is a mass term for the electron. This term appears even though the original Lagrangian is gauge invariant.

7.2 Fermion Interactions

The interactions of fermions with the W and Z bosons and photons is modelled in the Yukawa sector by

$$\bar{\psi} i \not{D} \psi = \bar{\psi} i \not{\partial} \psi - \bar{\psi}\gamma^\mu (g \sum_a W_\mu^a T^a + g' B_\mu Y)\psi \tag{7.3}$$

This second term can be written as the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -e \left\{ A_\mu j^\mu_{\text{EM}} + \frac{1}{\sin\theta_W \cos\theta_W} Z_\mu j^\mu_{\text{NC}} + \frac{1}{\sqrt{2}\sin\theta_W}(W^+_\mu j^\mu_{CC} + W^-_\mu {j^\mu_{CC}}^\dagger \right\} \tag{7.4}$$

where the electromagnetic, neutral, and charge currents are

$$\begin{aligned} j^\mu_{\text{EM}} &= \bar{\psi}\gamma^\mu (T^3 + Y)\psi \\ j^\mu_{\text{NC}} &= \bar{\psi}\gamma^\mu T^3 \psi - \sin^2\theta_W j^\mu_{\text{EM}} \\ j^\mu_{\text{CC}} &= \bar{\psi}\gamma^\mu (T^1 + iT^2)\psi. \end{aligned} \tag{7.5}$$

We can form $SU(2)_L$ singlets by combining the Higgs doublet with the left-handed fermion doublets and then combine these with the $SU(2)_R$ fermions. All possible combinations for the Yukawa Lagrangian can be written in terms of complex 3×3 matrices for the leptons and the up and down type quarks.

$$\mathcal{L} = -(\bar{\psi}_e\, \bar{\psi}_\mu\, \bar{\psi}_\tau)_R C_l \begin{pmatrix} \phi^\dagger \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \psi_{\nu_\mu} \\ \psi_\mu \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \psi_{\nu_\tau} \\ \psi_\tau \end{pmatrix}_L \end{pmatrix} - (\bar{\psi}_u\, \bar{\psi}_c\, \bar{\psi}_t)_R {C'}_q \begin{pmatrix} \widetilde{\phi}^\dagger \begin{pmatrix} \psi_u \\ \psi_{d'} \end{pmatrix}_L \\ \widetilde{\phi}^\dagger \begin{pmatrix} \psi_c \\ \psi_{s'} \end{pmatrix}_L \\ \widetilde{\phi}^\dagger \begin{pmatrix} \psi_t \\ \psi_{b'} \end{pmatrix}_L \end{pmatrix} - (\bar{\psi}_{d'}\, \bar{\psi}_{s'}\, \bar{\psi}_{b'})_R C_q \begin{pmatrix} \phi^\dagger \begin{pmatrix} \psi_u \\ \psi_{d'} \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \psi_c \\ \psi_{s'} \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \psi_t \\ \psi_{b'} \end{pmatrix}_L \end{pmatrix} + h.c. \tag{7.6}$$

This form of the Yukawa Lagrangian is invariant under $SU(2)_L \times U(1)_Y$ by construction. The primed fields allow for a mismatch between fields which couple to the gauge bosons and fields which are eigenstates of the mass matrix. We can reduce this Lagrangian to the purely physical parameters using symmetries. We find that we can reduce the lepton C_l matrix to a diagonal form where the diagonal entries are related to the lepton masses. We can also similarly diagonalise the ${C'}_q$ matrix.

$$C_l = \text{diag}(c_e, c_\mu, c_\tau), \quad {C'}_q = \text{diag}(c_u, c_c, c_t) \tag{7.7}$$

By diagonalising the up type quark matrix, we have run out of degrees of freedom to diagonalise the down type quark matrix. Instead we transform it to a standard form

$$C_q = V_{\text{CKM}} \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V^\dagger_{\text{CKM}}. \tag{7.8}$$

where V_{CKM} is the unitary $3\times$ Cabibbo-Kobayashi-Maskawa (CKM) matrix.

7.3 Quark Mixing

Looking at the Yukawa Lagrangian in the unitary gauge $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H \end{pmatrix}$, we have

$$\begin{aligned}\mathcal{L} = \frac{v}{\sqrt{2}}\left(1+\frac{H}{v}\right)\Bigg\{ &-(\bar{\psi}_e\ \bar{\psi}_\mu\ \bar{\psi}_\tau)_R \begin{pmatrix} c_e & 0 & 0 \\ 0 & c_\mu & 0 \\ 0 & 0 & c_\tau \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix}_L \\ &-(\bar{\psi}_u\ \bar{\psi}_c\ \bar{\psi}_t)_R \begin{pmatrix} c_u & 0 & 0 \\ 0 & c_c & 0 \\ 0 & 0 & c_t \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_c \\ \psi_t \end{pmatrix}_L \\ &-(\bar{\psi}_d\ \bar{\psi}_s\ \bar{\psi}_b)_R V_{\text{CKM}} \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V_{\text{CKM}}^\dagger \begin{pmatrix} \psi_d \\ \psi_s \\ \psi_b \end{pmatrix}_L + h.c.\Bigg\}.\end{aligned} \tag{7.9}$$

The mass terms are $m_e = c_e\frac{v}{\sqrt{2}}$, ..., $m_b = c_b\frac{v}{\sqrt{2}}$ and the mass eigenbasis is

$$\begin{pmatrix} \psi_d \\ \psi_s \\ \psi_b \end{pmatrix} = V_{\text{CKM}}^\dagger \begin{pmatrix} \psi_{d'} \\ \psi_{s'} \\ \psi_{b'} \end{pmatrix}. \tag{7.10}$$

As a consequence, the charge current becomes

$$j^\mu_{\text{CC}} = (\bar{\psi}_{\nu_e}\ \bar{\psi}_{\nu_\mu}\ \bar{\psi}_{\nu_\tau})_L\gamma^\mu \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix}_L + (\bar{\psi}_u\ \bar{\psi}_c\ \bar{\psi}_t)_L\gamma^\mu V_{\text{CKM}} \begin{pmatrix} \psi_d \\ \psi_s \\ \psi_b \end{pmatrix}_L. \tag{7.11}$$

The CKM term mixes quark generations and allow for transitions (weak decays) such as

$$\begin{aligned} d &\to u + W^- \\ s &\to u + W^- \\ b &\to u + W^-. \end{aligned} \tag{7.12}$$

The CKM matrix is related to the transition amplitudes and gives the transitions

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{7.13}$$

The current experimental measurements for these values are

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.97370 \pm 0.00014 & 0.2245 \pm 0.0008 & 0.00382 \pm 0.00024 \\ 0.221 \pm 0.004 & 0.987 \pm 0.011 & 0.0410 \pm 0.0014 \\ 0.0080 \pm 0.0003 & 0.0388 \pm 0.0011 & 1.013 \pm 0.030 \end{pmatrix} \tag{7.14}$$

The unitarity of the CKM matrix can be checked using these values and for example we find the up quark transition amplitudes are

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9985 \pm 0.0005 \tag{7.15}$$

which appears to violate unitarity to a 3-sigma confidence level. This offers a hint at physics beyond the Standard Model.

If we consider only two families[‡], we can parameterise the CKM matrix by the Cabibbo angle θ_c as

$$V^{(2)}_{\text{CKM}} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \tag{7.16}$$

[‡]This two-family case was used extensively in the 1970s with the discovery of the charm quark. The three-family case was needed to explain CP violation in the Weak interaction.

This is a real rotation matrix in the plane of the first two quark families. There is an analogous argument for the three family case where the physical parameters are three angles θ_{12}, θ_{13}, and θ_{23} and a complex phase δ.

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{7.17}$$

Here, $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. $0 \leq \delta \leq 2\pi$ is the phase.

7.4 CP Violation

In the Wolfenstein parametrisation (an expansion in powers of the Cabibbo angle), the CKM matrix is

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{7.18}$$

Measurement of $\rho - i\eta$ allows one to measure CP violation as it is related to the phase δ which measures the "strength" of CP violations in weak decays.

Transforming the Yukawa action under charge and parity transformations results in

$$\mathcal{S}_{\text{Yukawa}} \overset{\text{CP}}{\to} \mathcal{S}_{\text{Yukawa}}\big|_{V_{\text{CKM}} \to V_{\text{CKM}}{}^*} \tag{7.19}$$

We can see that if the CKM matrix is real, the action is CP-symmetric, since $V_{\text{CKM}} = V_{\text{CKM}}{}^*$. If the CKM matrix has a complex phase ($\delta \neq 0$), then this CP symmetry is violated.

The Nobel Prize-winning discovery of CP violations in kaon decays by Cronin and Fitch in the 1960s led to the theorisation of a third family of quarks. Two families are not enough to allow for CP violations in the Standard Model, as we saw the 2× Cabibbo matrix is purely real. Kobayashi and Maskawa won the 2008 Nobel Prize in Physics for their explanation of CP violation through the CKM matrix, however Cabibbo whose work was built on by Kobayashi and Maskawa, was omitted from the prize to the consternation of many physicists.

8 Beyond the Standard Model

8.1 Effective Field Theories

In early quantum field theories, it was realised that when computing probability amplitudes from Feynman diagrams, the loop terms would constitute an infinitely large correction to the amplitude when integrated over all of the intermediate virtual energies. For example, the loop in Fig. 1 contributes an integral over the intermediate energy E which can take any value from $-\infty$ to ∞. The original way to deal with these terms, as used by Feynman, Schwinger, Tomonaga, and Dyson, was to introduce a counter-term to "sweep the infinities under the rug".

In the 1970s the lesser-known physicist Ken Wilson came up with the idea of an effective field theory, which now forms the basis of most "modern" quantum field theories. This formalism argues that the reason these infinities arise in the loop Feynman diagrams is because the theory is considering arbitrarily high energies E, well outside of the scope of the theory itself. Recall that the energy behaves as

$$E \sim \frac{1}{\lambda} \tag{8.1}$$

so the high energy contributions correspond to very small distances - beyond the Planck scale - which, Wilson argued, falls outside of the scope of "everyday" particle physics. Quantum Field Theory does

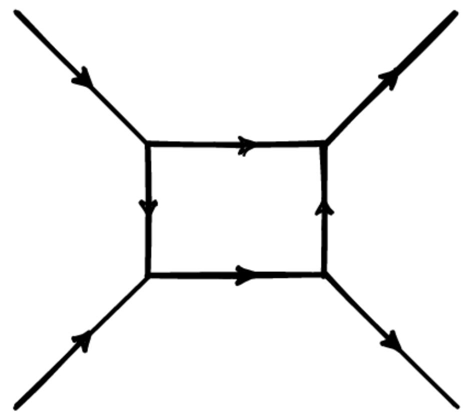

Figure 1: A loop Feynman diagram which gives an infinite contribution to the probability amplitude

not tells us what happens at these high energies and we should be able to do good physics without having to consider them. Instead of including these energies, Wilson introduced a cutoff energy E_* above which we swallow our pride and admit we don't know what's going on.

Can we justify this sort of "cutoff" of our theory? Yes, because the particles we discuss are much lower in energy. For example, the electron and proton have masses (in MeV) of

$$m_e = 0.511 \text{ MeV}, \quad m_p = 938 \text{ MeV} \tag{8.2}$$

while the Planck mass, the energy scale at which we expect quantum gravity to couple (string theory, etc.), is around $m_{\text{pl}} \sim 10^{21}$ MeV, many orders of magnitude higher than run-of-the-mill physics.

This is fine, but we can go further and consider the example of a free scalar field. The kinetic energy density of such a theory is $\frac{1}{2}\dot{\phi}^2$. The units of this quantity are

$$\left[\frac{1}{2}\dot{\phi}^2\right] = \frac{[E]}{[D]^3} = [E]^4, \tag{8.3}$$

since $[E] = [D]^{-1}$ in units of $c = \hbar = 1$. We then find the dimensions of the field itself

$$[E]^4 = [\dot{\phi}^2] = \frac{[\phi]^2}{[T]^2} = [\phi]^2[E^2] \implies [\phi] = [E]. \tag{8.4}$$

So, the scalar field ϕ has dimensions of energy.

Now, the action is the integral of the Lagrangian, which is itself an integral of the Lagrangian density

$$\mathcal{S} = \int L dt = \int \int \mathcal{L} d^3x dt \tag{8.5}$$

The units of the Lagrangian L are energy, since its definition is the difference in kinetic and potential energies (modulo some interaction terms). The differential d^3x has units of $[D]^3$, so the Lagrangian density must have units $[\mathcal{L}] = [E]^4$.

If we consider terms in the Lagrangian (I'll drop "density" from now on) of order ϕ^3, this corresponds to a Feynman diagram with three ϕ vertices. ϕ^4 corresponds to four scalar field vertices, and so on. Since the Lagrangian has units $[E]^4$, one might guess that only ϕ^4 terms can be in the Lagrangian. This is not the case however, as we can have coupling constants with units of energy. Once these combined units make up $[E]^4$, we're good

For example, the QED Lagrangian has interaction terms like

$$\mathcal{L}_{\text{QED}} \sim \sqrt{\alpha} e^- e^+ \gamma \tag{8.6}$$

for an electron, positron and photon. The electron and positron fields have dimension $[E]^{3/2}$ and the electromagnetic field has units of energy, so we naturally have dimensions of $[E]^4$ and the coupling constant α is dimensionless.

If we consider a general scalar Lagrangian with interaction terms

$$\mathcal{L} \sim c_3 \phi^3 + c_4 \phi^4 + c_5 \phi^5 + \cdots , \tag{8.7}$$

then because we want each term to have units of $[E]^4$ and ϕ has units of energy, we *must have*

$$[c_3] = [E], \quad [c_4] = 1, \quad [c_5] = \frac{1}{[E]}, \quad \text{etc.} \tag{8.8}$$

The most natural expectation for the values of these coupling constants, ignoring any "fine-tuning", is that they are of the order of the cutoff energy itself. That is, the coefficients should be in-or-around

$$c_3 \sim \mathcal{O}(E_*), \quad c_4 \sim \mathcal{O}(1), \quad c_5 \sim \mathcal{O}(1/E_*), \quad \text{etc..} \tag{8.9}$$

If we think of QED again, the appropriate coupling constant was $c_4 = \sqrt{\alpha}$. Roughly speaking, $\alpha \simeq \frac{1}{137}$, so $\sqrt{\alpha} \sim \frac{1}{10}$ is roughly of order one (within one order of magnitude).

Because the coupling constants c_5 and above are negative powers of the cutoff, we should be able to ignore them for a sufficiently high cutoff energy. They don't contribute to the theory. These are called "irrelevant" interactions, while ϕ^4 is a "marginal" interaction and ϕ^3 is a "relevant" interaction.

8.2 Testing EFTs & the Higgs

The payoff we get with using an effective theory up to some cutoff is that there are only a handful of relevant terms in the theory and we can ignore higher terms. Recall the Higgs Lagrangian (6.15) (without gauge boson interactions) is

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2} m_H^2 H^2 - \lambda v H^3 - \frac{\lambda}{4} H^4, \tag{8.10}$$

which has interaction terms only going up to the marginal H^4 term. Testing effective field theories can be done by performing experiments at the cutoff energy. If you can find a breakdown of an effective field theory at the cutoff energy, then you know that there is higher energy physics not described by your theory even though you haven't detected the new particles themselves.

The problem of course is if the new physics is at an energy way out of reach of your experiments. For example, we mentioned the quantum gravity scale of 10^{18} GeV, while the energies achievable by the Large Hadron Collider are a meagre 10^4 GeV in comparison. This is why it's essentially impossible to test string theory with our current technology.

Setting aside quantum gravity, a more pressing mystery in particle physics is that of the Higgs mass. Taking a look back up at the Higgs Lagrangian, the mass term is

$$\frac{1}{2} m_H^2 H^2 \tag{8.11}$$

We have verifiably measured the mass of the Higgs boson to be around $m_H \simeq 125$ GeV. If we had naïve trust in the Standard Model as the only theory up to quantum gravity, i.e. there is no new physics up until the Planck scale, then our cutoff should be *at the Planck scale* and we should see

$$m_H \sim E_* \sim 10^{18} \text{ GeV} \tag{8.12}$$

but we don't! The Higgs mass is 16 orders of magnitude smaller. This is the hierarchy problem; why is there such a massive gap between the scale of the electroweak (Higgs) scale and the Planck scale? If there is no new physics, we should have seen a much heavier Higgs boson. If there is new physics above 10^2 GeV, we should be able to turn on our particle accelertors and see it, but alas we don't see that either! At least, up to the 10^4 GeV of the LHC. So what's left for us to do? We don't know. This is as far as we have gotten. The rest remains to be seen.