MAU44404 - General Relativity
Brief Notes

Liam Kavanagh
B.A. Theoretical Physics

Oh leave the Wise our measures to collate
One thing at least is certain, light has weight
One thing is certain, and the rest debate—
Light rays, when near the Sun, do not go straight.

Sir Arthur S. Eddington, 1882-1944

1 Introduction

The material in this module closely follows *Lecture Notes on General Relativity* [1] by Sean M. Carroll. I'll be assuming a good knowledge of Differential Geometry and glossing over some common definitions. There is material covered in the course that is not covered here, mostly because they are asides to the main content, don't fit well into the structure of these notes, or are easily looked up from your own notes or on Wikipedia.

1.1 Inertial Frames

Consider two inertial frames $\mathcal{S}$ and $\mathcal{S}'$, with $\mathcal{S}'$ moving relative to $\mathcal{S}$ with a (dimensionless) velocity $\beta = v/c$ in the x direction; i.e. $\vec{\beta} = \beta\hat{x}$.

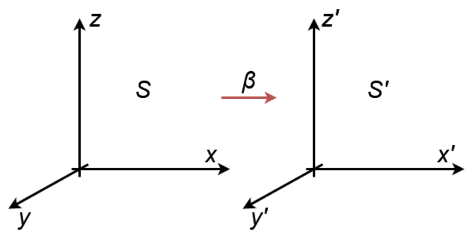


Figure 1: Two inertial frames related by a boost in the x direction

These frames are related by **Lorentz transformations (LTs)**. In units where $c = 1$ (as will be assumed from now on), the spatial and temporal coordinates transform like

$$\begin{aligned} x' &= \gamma(x - \beta t), \quad y' = y \\ t' &= \gamma(t - \beta x), \quad z' = z, \end{aligned} \tag{1.1}$$

where $\gamma = (1 - \beta^2)^{-1/2}$. Such LTs preserve the quantity*

$$s^2 = -t^2 + x^2 + y^2 + z^2 = -t'^2 + x'^2 + y'^2 + z'^2. \tag{1.2}$$

*This can also be equivalently defined with the opposite overall sign, but we will use the $- + ++$ convention.

If we consider just the tx plane, then we have the condition that a set of inertial frames have to satisfy

$$s^2 = -t^2 + x^2. \tag{1.3}$$

This is the equation of a hyperbola for some fixed constant s^2. Plotting the lines of constant s^2 in Fig. 2, we see that a particle moving in an inertial reference frame is constrained to one of the curves.

A particle on a blue curve ($s^2 < 0$) is time ordered with respect to the origin. That is to say, one cannot Lorentz transform a future event so that it happens in the past or vice versa. This region is causal, or **time-like**.

A particle on a red curve ($s^2 > 0$) is spatially ordered with respect to the origin and cannot be Lorentz transformed from "in front" of the origin to "behind" the origin, and vice versa however it can be boosted into the past or future and is therefore not causal. This region is called **space-like**.

A particle on a green curve ($s^2 = 0$) is moving along the lines $x = \pm t$, so its velocity is exactly that of light (here $c = 1$ so we have $x = \pm ct$). This region is called **light-like** or **null**.

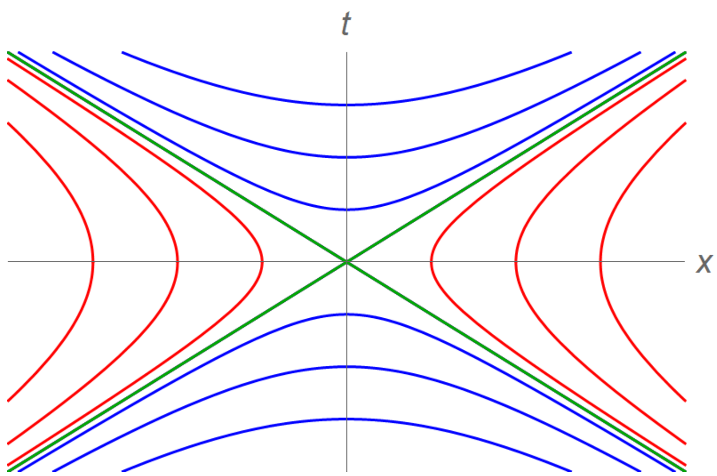


Figure 2: The equation $t^2 = x^2 - s^2$.

Since the quantity s^2 is invariant under Lorentz transformations, an unaccelerated particle is confined to its hyperbola.

1.2 Four-Vectors and the Lorentz Group

We can group the three spatial components and one temporal component into a single **four-vector** x^μ, where $\mu = 0, 1, 2, 3$. Namely,

$$x^\mu = (t, x, y, z) = (x^0, x^1, x^2, x^3). \tag{1.4}$$

The invariant quantity s^2 can then be written as

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu \tag{1.5}$$

Where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is a diagonal matrix called the **Minkowski metric** and pairs of lower and upper indices are summed over.

With this constraint, we can ask what is the most general Lorentz transformation we can perform? Consider the transformation

$$x^\mu \to \ell^\mu_\nu x^\nu. \tag{1.6}$$

The invariant quantity s^2 is then

$$s^2 = \eta_{\alpha\beta} x^\alpha x^\beta = \eta_{\mu\nu} \ell^\mu_\alpha x^\alpha \ell^\nu_\beta x^\beta. \tag{1.7}$$

Thus we require $\eta_{\alpha\beta} = \eta_{\mu\nu} \ell^\mu_\alpha \ell^\nu_\beta$, which in matrix notation is $\eta = \ell^T \eta \ell$. This gives several constraints on the transformation matrix ℓ. Taking the determinant, we get

$$\begin{aligned} \det \eta &= \det\left(\ell^T \eta \ell\right) \\ &= \det \eta (\det \ell)^2 \\ 1 &= (\det \ell)^2 \\ \implies \det \ell &= \pm 1. \end{aligned} \tag{1.8}$$

"Proper" Lorentz transformations have $\det \ell = +1$ and "improper" (discrete) LTs have $\det \ell = -1$. We can also constrain the 00 component of ℓ by considering its block form. With

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \quad \ell = \begin{bmatrix} \ell_{00} & -\mathbf{a}^T \\ -\mathbf{b} & \mathbf{L} \end{bmatrix}, \tag{1.9}$$

we have the condition

$$\begin{aligned} \ell^T \eta \ell &= \begin{bmatrix} \ell_{00} & -\mathbf{b}^T \\ -\mathbf{a} & \mathbf{L}^T \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \ell_{00} & -\mathbf{a}^T \\ -\mathbf{b} & \mathbf{L} \end{bmatrix} \\ &= \begin{bmatrix} -\ell_{00}^2 + \mathbf{b}^T \mathbf{b} & \mathbf{a}^T \ell_{00} - \mathbf{b}^T \mathbf{L} \\ \mathbf{a} \ell_{00} - \mathbf{b} \mathbf{L}^T & \mathbf{L}^T \mathbf{L} - \mathbf{a}^T \mathbf{a} \end{bmatrix} \end{aligned} \tag{1.10}$$

So we have the condition that the 00 component must satisfy

$$-1 = -\ell_{00}^2 + \mathbf{b}^T \mathbf{b}. \tag{1.11}$$

Since $\mathbf{b}^T \mathbf{b}$ is strictly non-negative (it's the usual dot product), we have $\ell_{00}^2 \geq 1$, which gives two conditions: $\ell_{00} \geq 1$ or $\ell_{00} \leq -1$. Since the 00 component multiplies the time coordinate, the positive condition preserves the time direction ("orthochronous") and the negative condition is time reversal ("antichronous"). This gives four groups of Lorentz transformations:

	Orthochronous: $\mathcal{L}^\uparrow$ ($\ell_{00} \geq 1$)	Antichronous: $\mathcal{L}^\downarrow$ ($\ell_{00} \leq -1$)
Proper: $\mathcal{L}_+$ ($\det \ell = +1$)	$\mathcal{L}^\uparrow_+$	$\mathcal{L}^\downarrow_+$
Improper: $\mathcal{L}_-$ ($\det \ell = -1$)	$\mathcal{L}^\uparrow_-$	$\mathcal{L}^\downarrow_-$

We'll focus on the proper orthochronous subgroup $\mathcal{L}^\uparrow_+$.

1.3 Proper Time and Four-Velocity

If we consider a displacement

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = -\Delta t^2 + \Delta x^2, \tag{1.12}$$

we can transform to another coordinate system where the spatial displacement $\Delta x' = 0$ and the time interval is $\Delta t' := \Delta\tau$. This time interval is called the **proper time** interval, $\Delta\tau$. In terms of the original coordinates, it is

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2. \tag{1.13}$$

The proper time is the time measured by a clock travelling with the observer along the path. Notice that in the proper time frame the spacetime interval is

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = -\Delta\tau^2 \leq 0. \tag{1.14}$$

Thus, the proper time cannot be used to parametrise spacelike paths. For null paths, where $s^2 = 0$, the proper time is zero. i.e. light takes "zero proper time" to travel through space.

There is a simple relation between the proper time τ and the time t measured by some external observer.

$$\begin{aligned}
d\tau^2 &= dt^2 - dx^2 \\
&= dt^2 \left[1 - \left(\frac{dx}{dt} \right)^2 \right] \\
&= dt^2 (1 - \beta^2) \\
\implies d\tau &= dt\sqrt{1-\beta^2} \\
\implies \frac{dt}{d\tau} &= \gamma. \qquad (1.15)
\end{aligned}$$

We can use the proper time to define the **four-velocity**

$$u^\mu := \frac{dx^\mu}{d\tau}. \qquad (1.16)$$

We can see from this definition that the four-velocity is normalised to -1.

$$\begin{aligned}
u_\mu u^\mu &= \eta_{\mu\nu} u^\mu u^\nu \\
&= \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \\
&= (-1) \left(\frac{dt}{d\tau} \right)^2 + (1) \left(\frac{d\mathbf{x}}{d\tau} \right)^2 \\
&= - \left(\frac{dt}{d\tau} \right)^2 \left[1 - \left(\frac{d\mathbf{x}}{d\tau} \right)^2 \left(\frac{d\tau}{dt} \right)^2 \right] \\
&= -\gamma^2 \left[1 - \left(\frac{d\mathbf{x}}{dt} \right)^2 \right] \\
&= -\frac{1}{1-\beta^2}(1-\beta^2) \\
&= -1. \qquad (1.17)
\end{aligned}$$

We are always able to move to a rest frame where the particle is not moving with respect to the origin. In that case, the four-velocity is

$$u^0 = 1, \quad u^i = 0. \qquad (1.18)$$

1.4 The Twin Paradox

An interesting consequence of proper time is the so-called "Twin Paradox". Suppose there is a pair of identical twins; Alice and Bob. At some time, both twins are at the same position in space. Then, Bob takes off in a rocket with speed β (Fig. 3). After a while he turns around and comes back, taking the same amount of time both ways.

The proper time as measured by Alice (along the blue path) is simply $\tau_A = 2T$. As measured by Bob (along the red path), it's given by the Lorentz transformation

$$\tau_B = 2\gamma(T - \beta(\beta T)) = \frac{1-\beta^2}{\sqrt{1-\beta^2}} T = 2\sqrt{1-\beta^2}T \qquad (1.19)$$

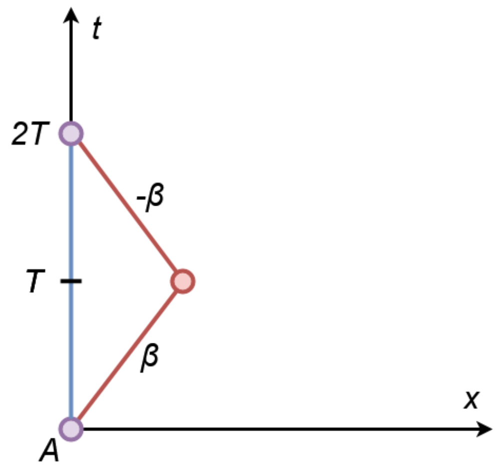


Figure 3: Setup for the Twin Paradox

So, we see that $\frac{\tau_A}{\tau_B} = \frac{2T}{\sqrt{1-\beta^2}2T} = \gamma$. Since $\gamma \geq 1$, Alice ages *faster* than Bob by a factor of γ.

To give an idea of how small this effect is in everyday life, when astronauts Mark Kelly and Mikhail Kornienko were on the International Space Station, they spent 340 days travelling at about 7,500 m/s relative to the Earth's surface. We thus have a Lorentz factor

$$\gamma = \left[1 - \left(\frac{7.5 \times 10^3}{3 \times 10^8}\right)^2\right]^{-1/2} = 1 + 3.125 \times 10^{-10}. \tag{1.20}$$

Over 340 days, this equates to someone on Earth ageing by an additional†

$$(3.125 \times 10^{-10})(60)^2(24)(340) \simeq 9 \text{ milliseconds}. \tag{1.21}$$

2 Examples of Curved Spacetime

2.1 Warp Drives

In an effort to rope you in with Sci-Fi, we'll start with a warp drive spacetime. Consider the metric

$$ds^2 = -dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2. \tag{2.1}$$

Here, $v_s = \frac{dx_s}{dt}$ is the slope of a spacelike trajectory $x_s(t)$. $r_s^2 = (x - x_s(t))^2 + y^2 + z^2$ is the spatial distance, and f is a smooth function of r_s such that $f(0) = 1$ and $f(R) = 0$, where $R << x$ is a cylindrical radius around $x_s(t)$.

Due to the function $f(r_s)$, the spacetime outside the cylinder is just Minkowski space and light cones points along the t axis. As a result, a timelike particle would not be able to access the spacelike trajectory. Inside the cylinder however, we have a different spacetime. We can find the light cone by

†There is actually an additional factor at play here; the gravitational time dilation due to the presence of a gravitational field. A gravitational potential actually makes the twin further from the Earth's surface age faster, lessening the effects of the additional speed!

computing $ds^2 = 0$. On the trajectory $x_s(t)$, we have $f(0) = 1$, so

$$\begin{aligned} ds^2 = 0 &= -dt^2 + (dx - v_s dt)^2 \\ \implies dt^2 &= (dx - v_s dt)^2 \\ dx - v_s dt &= \pm dt \\ \frac{dx}{dt} &= v_s \pm 1 \\ &= \frac{dx_s}{dt} \pm 1. \end{aligned} \tag{2.2}$$

The light cone points along the trajectory $x_s(t)$, so the spacelike trajectory is accessible to a timelike particle entering the trajectory at the origin.

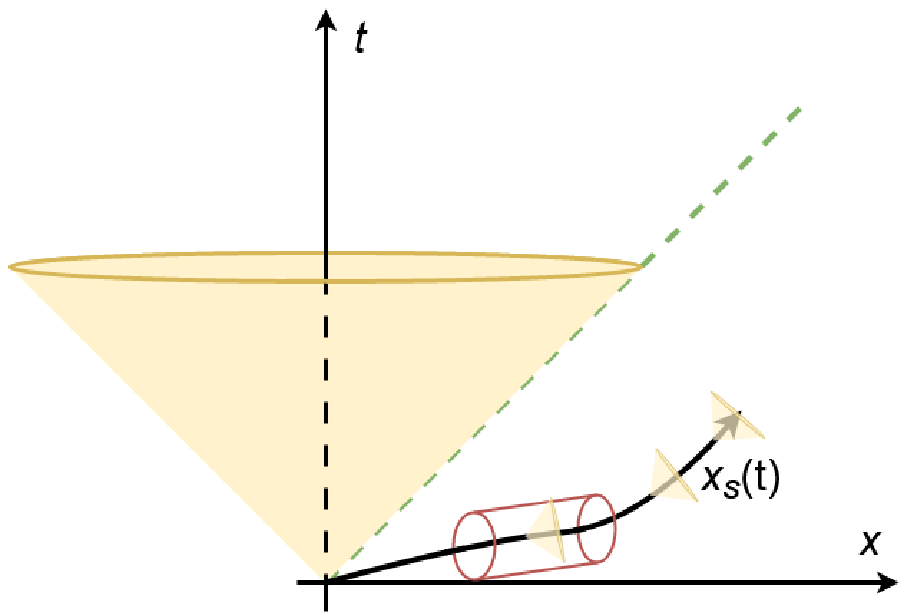


Figure 4: The warp drive spacetime.

This allows a timelike particle to "travel" at and above the speed of light, however such a spacetime requires a negative energy density.

2.2 Wormholes

In yet another cheap attempt at keeping your attention, we'll now consider a traversible wormhole. Consider the $2+1$ dimensional metric

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)d\phi^2, \quad \phi \in [0, 2\pi]. \tag{2.3}$$

We see that if $b = 0$, we just get flat Minkowski space in polar coordinates, and for $r >> b$ we also have flat space, since $r^2 + b^2 \simeq r^2$. We can embed this metric into $3+1$ dimensional space in cylindrical coordinates:

$$ds^2_{\text{cyl}} = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2, \quad \phi \in [0, 2\pi]. \tag{2.4}$$

We can rewrite this cylindrical metric in terms of the dependence on ρ, namely

$$ds^2_{\text{cyl}} = -dt^2 + \left(\frac{dz}{d\rho}\right)^2 d\rho^2 + d\rho^2 + \rho^2 d\phi^2. \tag{2.5}$$

To match these two metrics, we need the functions $\rho(r)$ and $z(r)$, both in terms of r. We can write the metric in terms of r as

$$\begin{aligned} ds^2_{\text{cyl}} &= -dt^2 + \left(\frac{d\rho}{dr}\right)^2 \left[\left(\frac{dz}{d\rho}\right)^2 + 1\right] dr^2 + \rho^2 d\phi^2 \\ &= -dt^2 + \left[\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2\right] dr^2 + \rho^2 d\phi^2. \end{aligned} \tag{2.6}$$

Matching coefficients with (2.3), we see immediately from the $d\phi^2$ term that

$$\rho^2 = r^2 + b^2 \implies \rho = \sqrt{r^2 + b^2}, \quad d\rho = \frac{r}{\sqrt{r^2 + b^2}} dr. \tag{2.7}$$

Using this on the dr^2 term, we find an equation for $z(r)$

$$\left(\frac{dz}{dr}\right)^2 + \frac{r^2}{r^2 + b^2} = 1. \tag{2.8}$$

The solution to this equation is

$$z(r) = \pm b \ln \left(\frac{r + \sqrt{r^2 + b^2}}{C}\right) \tag{2.9}$$

for some integration constant C. We can fix $C = b$ by imposing $z(r = 0) = b$.

This spacetime is called an Ellis wormhole, and connects two copies of Minkowski space together via a “throat” at the origin.

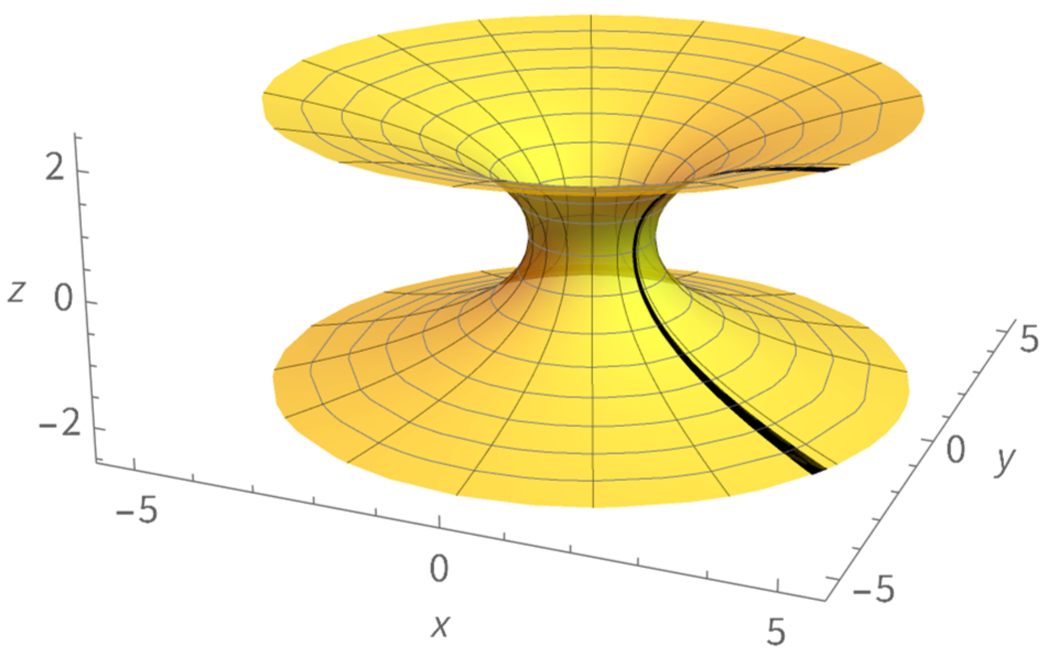


Figure 5: An Ellis wormhole, with one possible geodesic path marked in black.

3 Linearised Gravity

3.1 The Equivalence Principle

The Einstein Equivalence Principle (EEP) asserts that in small enough regions of spacetime, physics reduces to that of Special Relativity. As a result, it's not possible to differentiate between a uniformly accelerating reference frame and a gravitational field.

In terms of Riemannian geometry, this statement is the same as the ability to find Riemann Normal Coordinates at each point on a manifold. The geodesic equation, for example,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{3.1}$$

reduces (in RNCs, where the connection vanishes) to $\dfrac{d^2x^\mu}{d\tau^2} = 0$, which is the equation for a line.

When we do have a gravitational field, we'd want to hope that the geodesic equation above gives the correct answer in the non-relativistic limit. That is, it reduces to Newton's law of gravitation

$$\mathbf{a}_g = -\nabla\Phi = -\frac{GM}{r^2}. \tag{3.2}$$

This Newtonian approximation requires that we have both slow moving particles and a weak, static gravitational field. The requirement of the particles being slow-moving is

$$\frac{dx^i}{d\tau} << \frac{dx^0}{d\tau} = \frac{dt}{d\tau}. \tag{3.3}$$

Thus, neglecting all except the $\rho = 0, \sigma = 0$ terms, the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{3.4}$$

Since the field is static, the time derivatives in the Christoffel symbols vanish:

$$\begin{aligned}\Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00})\\ &= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}.\end{aligned} \tag{3.5}$$

Decomposing the weak-field metric into the Minkowski metric, $\eta_{\mu\nu}$, plus a perturbation yields

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1. \tag{3.6}$$

The inverse metric definition $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$ gives that

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.7}$$

We thus have for the Christoffel components

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}, \tag{3.8}$$

which simplifies the geodesic equation to

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2. \tag{3.9}$$

Noticing that for $\mu = 0$ this equation simply gives $\frac{d^2t}{d\tau^2} = 0$, we find that $\frac{dt}{d\tau}$ is constant. Since the spacelike components of the Minkowski metric are just 1s, we can write the rest of the geodesic equation as

$$\begin{aligned} \frac{d^2x^i}{d\tau^2} &= \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00}, \\ \implies \frac{d^2x^i}{dt^2} &= \frac{1}{2}\partial_i h_{00}. \end{aligned} \tag{3.10}$$

If we let $h_{00} = -2\Phi$, this equation is exactly Newton's gravitational law:

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{a} = -\nabla\Phi. \tag{3.11}$$

Identifying h_{00} in this way means the metric has the component

$$g_{00} = -(1+2\Phi) = -\left(1 - \frac{2GM}{r}\right). \tag{3.12}$$

3.2 Gauge Invariance

As previously, the weak gravity condition is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1. \tag{3.13}$$

In this regime, the Christoffel connection becomes

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda}\partial_\lambda h_{\mu\nu}). \tag{3.14}$$

The Riemann and Ricci tensors are (to first order)

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\right) \\ R_{\mu\nu} &= \partial_\mu\partial_\nu h^{\mu\nu} - \partial_\mu\partial^\mu(\eta^{\mu\nu}h_{\mu\nu}) = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h, \end{aligned} \tag{3.15}$$

where here $h := \eta^{\mu\nu}h_{\mu\nu}$ and $\Box = \partial_\mu\partial^\mu$. The Einstein tensor becomes

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ &= \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\Box h\right). \end{aligned} \tag{3.16}$$

There is a gauge freedom in the perturbation $h_{\mu\nu}$ due to the fact that in different coordinate systems, the perturbation $h_{\mu\nu}$ may have a different form. The equivalent perturbations can be considered a set of diffeomorphisms leaving $h_{\mu\nu}$ small. In general, we would have, for sufficiently small ε,

$$h'_{\mu\nu} = h_{\mu\nu} + \varepsilon\mathcal{L}_\xi g_{\mu\nu} \tag{3.17}$$

where ξ is the vector field generating the diffeomorphism. Since the Lie derivative of the metric is zero, this can be written as

$$h'_{\mu\nu} = h_{\mu\nu} + \varepsilon(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu). \tag{3.18}$$

One such choice of gauge is the Lorenz gauge

$$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h = 0. \tag{3.19}$$

If we let $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, then the Lorenz gauge condition becomes

$$\partial_\mu \bar{h}^\mu_\nu = \partial^\mu \bar{h}_{\mu\nu} = 0. \tag{3.20}$$

3.3 Linearised Field Equations

With the Lorenz gauge condition and the linearised Einstein tensor, we can find the linearised Einstein field equations. Note that taking the trace of $\bar{h}_{\mu\nu}$ (in $3+1$ dimensions) gives

$$\bar{h} = h - \frac{1}{2}4h = -h \tag{3.21}$$

The perturbation can then be written as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{3.22}$$

Putting this into the Einstein tensor, we get

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} + \partial^\alpha\partial_{(\mu}\bar{h}_{\nu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta}. \tag{3.23}$$

Using the Lorenz gauge condition (3.20), we get simply

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}. \tag{3.24}$$

The linearised Einstein field equations are then

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{3.25}$$

In the static case, these equations are solvable. We find

$$\nabla^2\bar{h}_{00} = -16\pi G T_{00} \tag{3.26}$$

which has a solution in terms of the gravitational potential

$$\bar{h}_{00} = -4\Phi. \tag{3.27}$$

Moving back to the original perturbation, we recover $h_{00} = -2\Phi$ as before. We additionally find that $h_{ij} = 0$ for $i \neq j$ and $h_{ii} = -2\Phi$. The full linearised metric is then

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)d\mathbf{x}^2. \tag{3.28}$$

3.4 Gravitational Radiation

We can now consider the example of a gravitating system emitting gravitational waves in the linearised regime. This is very similar to moving radiating charges in electromagnetism. We have the linearised field equations

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{3.29}$$

The solution to this equation is given in terms of the retarded Green's function $G(x,y)^\ddagger$ defined by

$$\Box G(x,y) = \delta^{(4)}(x-y) \implies G(x,y) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\Theta(x^0-y^0)\delta(|\mathbf{x}-\mathbf{y}|-(x^0-y^0)). \tag{3.30}$$

This gives the solution

$$\bar{h}_{\mu\nu} = -16\pi G\int d^3y G(x-y)T_{\mu\nu}(y) = 4G\int d^3y\frac{T_{\mu\nu}(t_R,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}, \tag{3.31}$$

‡Not to be confused with Newton's constant, G, or the Einstein tensor $G_{\mu\nu}$. There are not enough letters...

where $t_R = t - |\mathbf{x}-\mathbf{y}|$ is the retarded time.

Performing a Fourier transform in t, we find

$$\begin{aligned}
\widetilde{\bar{h}}_{\mu\nu} &= \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\bar{h}_{\mu\nu}(t,\mathbf{x}) \\
&= \frac{4G}{\sqrt{2\pi}}\int dt_R d^3y e^{i\omega t_R + i\omega|\mathbf{x}-\mathbf{y}|}\frac{T_{\mu\nu}(t_R,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\
&= 4G\int d^3y e^{i\omega|\mathbf{x}-\mathbf{y}|}\frac{\widetilde{T}_{\mu\nu}(\omega,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}.
\end{aligned} \tag{3.32}$$

Here, $\widetilde{T}_{\mu\nu}$ is the Fourier transform of the stress-energy tensor. If we consider a slow moving far field such that $r \simeq |\mathbf{x}-\mathbf{y}|$ is constant and much larger than the radius of the source, we have

$$\widetilde{\bar{h}}_{\mu\nu} = 4G\frac{e^{i\omega r}}{r}\int d^3y \widetilde{T}_{\mu\nu}(\omega,\mathbf{y}). \tag{3.33}$$

Recalling the condition $\partial_\mu \widetilde{\bar{h}}_{\mu\nu} = 0$, we get from the above $\widetilde{\bar{h}}^{0\nu} = \frac{i}{\omega}\partial_i \widetilde{\bar{h}}^{i\nu}$. Then,

$$\begin{aligned}
\int d^3y \widetilde{T}^{ij} &= \int d^3y \left[\partial_k(y^i \widetilde{T}^{kj} - y^j\partial_k \widetilde{T}^{kj}\right] \\
&= i\omega \int d^3y\ y^j \widetilde{T}^{0j} \\
&= \frac{i\omega}{2}\int d^3y \left(y^i\widetilde{T}^{0j} + y^i\widetilde{T}^{0i}\right) \\
&= \frac{i\omega}{2}\int d^3y\ \partial_l\left(y^iy^j\widetilde{T}^{0l}\right) - y^iy^j\partial_l\widetilde{T}^{0l} \\
&= -\frac{\omega^2}{2}\int d^3y\ y^iy^j\widetilde{T}^{00}.
\end{aligned} \tag{3.34}$$

This integral is the Fourier transform of the quadrupole moment

$$q^{ij} = 3\int d^3y\ y^iy^jT^{00}(t,\mathbf{y}). \tag{3.35}$$

Thus, we have

$$\widetilde{\bar{h}}_{ij}(\omega,\mathbf{x}) = -\frac{2G\omega^2}{3}\frac{e^{i\omega r}}{r}\widetilde{q}_{ij} \implies \bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{3r}\left.\frac{d^2q_{ij}}{dt^2}\right|_{t_R}. \tag{3.36}$$

At a second order expansion, the Einstein tensor is proportional to an effective stress-energy tensor

$$t_{\mu\nu} = \frac{1}{32\pi G}\partial_\mu h^\alpha_\beta\partial_\nu h^\beta_\alpha. \tag{3.37}$$

From this we can find the energy loss over time due to gravitational radiation.

$$-\frac{d\mathcal{E}}{dt} = \frac{G}{45}\dddot{Q}_{ij}\dddot{Q}^{ij}\Big|_{t_R}, \quad Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}q_{kl}. \tag{3.38}$$

In the case of a binary star system with equal mass stars separated by a diameter $2r$, the frequency of rotation is $\Omega = \frac{1}{r}\sqrt{\frac{GM}{4r}}$. The gravitational charge is in terms of the mass distribution T^{00} =M$\delta^{(3)}(\mathbf{x}-\mathbf{x}_{\text{star}})$ as

$$Q_{ij} = Mr^2\begin{pmatrix} 3\cos 2\Omega t + 1 & 3\sin 2\Omega t \\ 3\sin 2\Omega t & -3\cos 2\Omega t + 1\end{pmatrix}, \quad \dddot{Q}_{ij} = 12Mr^2\Omega^3\begin{pmatrix}\sin 2\Omega t & -\cos 2\Omega t \\ -\cos 2\Omega t & -\sin 2\Omega t\end{pmatrix}. \tag{3.39}$$

The radiated power is then

$$-\frac{d\mathcal{E}}{dt} = \frac{128G}{5}M^2r^4\Omega^6. \tag{3.40}$$

4 Static, Spherically Symmetric Spacetimes

4.1 Circular Orbits

The metric for a static, spherically symmetric spacetime is the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where $2GM := R_s$ is the Schwarzschild radius and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit 2-sphere. For a timelike geodesic parametrised by the proper time τ, the geodesic equation decomposes into four coupled equations.

$$\begin{aligned}
&\frac{d^2t}{d\tau^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0\\
&\frac{d^2r}{d\tau^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\tau}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\tau}\right)^2 - (r-2GM)\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] = 0\\
&\frac{d^2\theta}{d\tau^2} + 2\frac{d\theta}{d\tau}\frac{dr}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0\\
&\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} + \frac{2\cos\theta}{\sin\theta}\frac{d\theta}{d\tau}\frac{d\phi}{d\theta} = 0.
\end{aligned} \tag{4.2}$$

Because we have spherical symmetry, we can consider an equatorial geodesic with $\theta = \frac{\pi}{2}$ which reduces these equations to

$$\begin{aligned}
&\frac{d^2t}{d\tau^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0\\
&\frac{d^2r}{d\tau^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\tau}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\tau}\right)^2 - (r-2GM)\left(\frac{d\phi}{d\tau}\right)^2 = 0\\
&\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} = 0.
\end{aligned} \tag{4.3}$$

The remaining symmetries correspond to the killing vectors

$$\begin{aligned}
k^\mu &= (1,0,0,0)\\
R^\mu &= (0,0,0,1).
\end{aligned} \tag{4.4}$$

The energy is given by

$$E = -k_\mu\frac{dx^\mu}{d\tau} = -g_{\mu\nu}k^\nu\frac{dx^\mu}{d\tau} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}. \tag{4.5}$$

As $r \to \infty$, $E \simeq \frac{dt}{d\tau}$ so for timelike geodesics we have $E \simeq \gamma$, i.e. the energy per unit mass.

The angular momentum is

$$L = R_\mu\frac{dx^\mu}{d\tau} = r^2\frac{d\phi}{d\tau}. \tag{4.6}$$

The normalisation condition for the four-velocity can be written as

$$\varepsilon = -g_{\mu\nu}u^\mu u^\nu, \tag{4.7}$$

where $\varepsilon = 1$ for timelike curves and $\varepsilon = 0$ for null curves. Expanding this condition gives

$$\begin{aligned}
\varepsilon &= \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2 \\
\left(1 - \frac{2GM}{r}\right)\varepsilon &= \left(1 - \frac{2GM}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2\left(1 - \frac{2GM}{r}\right)\left(\frac{d\phi}{d\tau}\right)^2 \\
&= E^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{r^2}\left(1 - \frac{2GM}{r}\right) \\
\frac{1}{2}E^2 &= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r)
\end{aligned} \tag{4.8}$$

where $V_{\text{eff}}(r) = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \varepsilon\right)$ is the effective potential.

In the timelike case ($\varepsilon = 1$), we get

$$V_{\text{eff}}(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{4.9}$$

The first three terms constitute the usual effective Newtonian potential for circular motion, and the last term is a correction of General Relativity. This potential is plotted in Fig. 6 for different values of GM and L.

The maxima and minima of the potential are given by

$$\frac{dV_{\text{eff}}}{dr} = \frac{1}{r^4}\left[GMr^2 - L^2 r + 3GML^2\right] = 0. \tag{4.10}$$

The solutions are

$$R_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2G^2M^2}}{2GM}. \tag{4.11}$$

Whenever $L^2 < 12G^2M^2$ there are no extrema, as seen in Fig. 6 (a). To have a circular orbit, we much have $L^2 > 12G^2M^2$, as in Fig. 6 (b). This means stable circular orbits occur at $R_+ > 6GM$, and $r = 6GM$ is the smallest stable circular orbit.

With unstable orbits, as $L \to \infty$ we find

$$R_- \simeq \frac{L^2 - L^2(1 - \frac{12G^2M^2}{L^2})^{1/2}}{2GM} = 3GM, \tag{4.12}$$

so the smallest unstable circular orbit is at $r = 3GM$ (from above), as in Fig. 6 (c).

The condition for a circular orbit, (4.10), gives us an expression for the angular momentum

$$L^2 = \frac{GMR^2}{R - 3GM}. \tag{4.13}$$

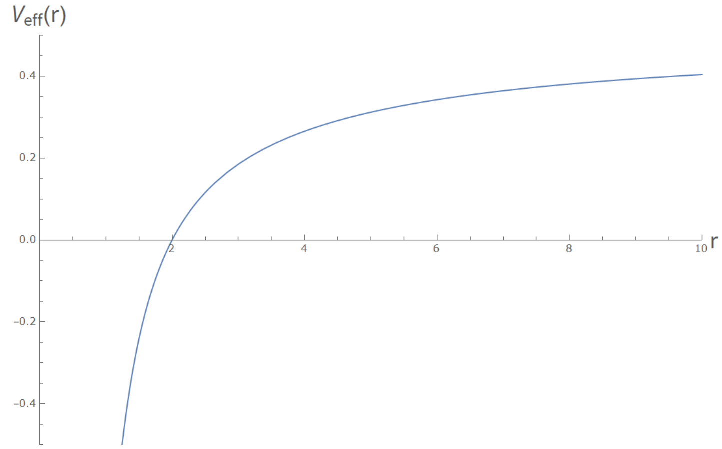
Substituting this into the energy equation (4.8), we get

$$\frac{1}{2}E^2 = \frac{1}{2R}\frac{(R - 2GM)^2}{R - 3GM} \implies E(R) = \frac{R - 2GM}{\sqrt{R}\sqrt{R - 3GM}}. \tag{4.14}$$

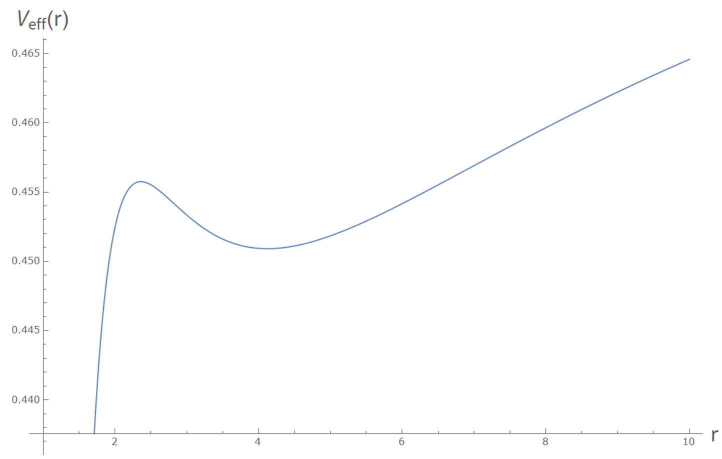
For $R < 4GM$, $E(R) > 1$ so the particle can escape to infinity. At the last stable orbit, $R = 6GM$, we find

$$E = \frac{2\sqrt{2}}{3} < 1, \tag{4.15}$$

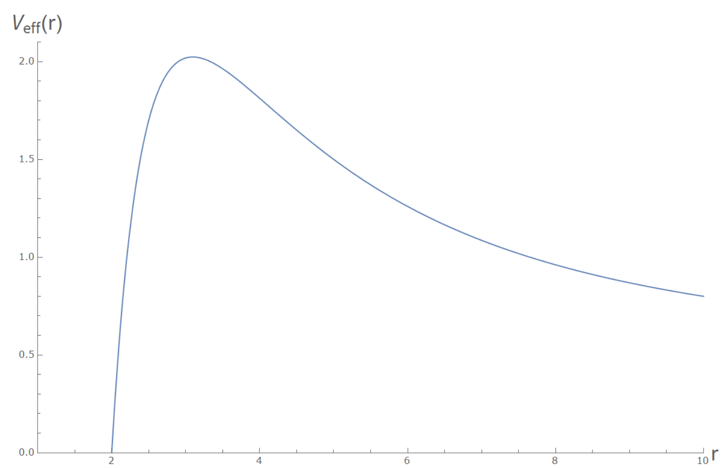
so the bound energy is $1 - \frac{2\sqrt{2}}{3} \simeq 0.06$. A particle spiralling from infinity radiates out about 6% of its energy.



(a) $V_{\text{eff}}(r)$ for $GM = 1$ and $L = 1$.



(b) $V_{\text{eff}}(r)$ for $GM = 0.5$ and $L = 1.8$.



(c) $V_{\text{eff}}(r)$ for $GM = 1$ and $L = 10$.

Figure 6

4.2 Orbital Precession

A particle in the last stable circular orbit ($R = 6GM$) will oscillate around the stable minimum if perturbed. In the radial direction, this oscillation is

$$\begin{aligned} \omega_r^2 &= \left.\frac{d^2 V_{\text{eff}}}{dr^2}\right|_{R_+} \\ &= \frac{1}{R_+^4}(2GMR_+ - L^2) \\ &= \frac{1}{R_+}\frac{GM(R_+ - 6GM)}{R_+ - 3GM}. \end{aligned} \tag{4.16}$$

Supposing that $R_+ << GM$, we see that

$$\omega_r^2 \simeq \frac{GM}{R_+^3}. \tag{4.17}$$

The radial frequency can also be computed as

$$\omega_\phi^2 = \frac{L^2}{R_+^4} = \frac{GM}{R_+^2(R_+ - GM)} \tag{4.18}$$

To leading order, this is again $\omega_\phi^2 \simeq \dfrac{GM}{R_+^3}$. To leading order, the orbit is closed but the subleading terms lead to an orbital precession

$$\omega_p = \omega_\phi - \omega_r = \omega_\phi \left[1 - \sqrt{1 - \tfrac{GM}{R_+}}\right] \simeq \omega_\phi \frac{3GM}{R_+} = \frac{3(GM)^{3/2}}{R^{5/2}}. \tag{4.19}$$

This precession of the perihelion of Mercury is one of the classical tests of General Relativity, since Newtonian mechanics cannot account for the experimentally measured precession. Corrections due to GR agree (almost) perfectly with the observed value.

4.3 Scattering in Spherical Spacetime

As in classical mechanics, a particle moving past a central mass will have its trajectory deflected. The motion of such a particle is described by the equation we found from the normalisation condition, (4.8), namely

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}E^2, \quad V_{\text{eff}}(r) = \frac{L^2}{r^2}\left(\frac{1}{2} - \frac{GM}{r}\right). \tag{4.20}$$

If the central mass is removed, i.e. $M = 0$, we expect that the particle's trajectory will be undisturbed. In this case, we have

$$V_{\text{eff}}(r)|_{M=0, r=b} = \frac{L^2}{2b^2} = \frac{E^2}{2}, \tag{4.21}$$

where here $r = b$ is the impact parameter. We see that this impact parameter is given by $b = \dfrac{L}{E}$. Using the definition of the angular momentum, we have

$$L = r^2\dot{\phi} \implies \frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \frac{L}{r^2}\frac{1}{\sqrt{E^2 - \frac{L^2}{r^3}(r - 2GM)}}. \tag{4.22}$$

The total angular deflection is the integral

$$\Delta\phi = 2\int_{r_0}^{\infty} \frac{b}{r^2}\frac{dr}{\sqrt{1 - \frac{b^2}{r^2}(r - 2GM)}} \tag{4.23}$$

where here r_0 is the minimum of the trajectory, given by the solution to $\dot{r} = 0$. If $M = 0$, then we have $r_0 = b$ and the integral is computable using the substitution $u = 1/r$.

$$\Delta\phi|_{M=0} = \int_0^{1/r_0} \frac{du}{\sqrt{r_0^{-2} - u^2}} = \pi. \tag{4.24}$$

Making a far-field (or weak mass) approximation $b >> GM$, we can expand the integral in powers of M. This gives a first-order correction

$$\Delta\phi = \pi + \frac{4GM}{r_0} + \mathcal{O}(M^2). \tag{4.25}$$

In the relativistic case, there is not just angular scattering but also "scattering through time"; the so-called Shapiro time delay. If we consider free space, the time taken for a particle to move from some point R_1 to R_0 is simply

$$\Delta t = \sqrt{R_1^2 - R_0^2}. \tag{4.26}$$

In the presence of a mass, we have (similar to the ϕ case)

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \left[\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)\right)\right]^{-1}. \tag{4.27}$$

So the total time is

$$T = \int_{r_0}^{r_1} \frac{1}{(1 - \frac{2GM}{r})} \frac{dr}{\sqrt{1 - \frac{b^2}{r^2}(1 - \frac{2GM}{r})}}. \tag{4.28}$$

The first correction at $\mathcal{O}(M)$ in the weak field approximation is

$$T = 2GM \ln\left(\frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_0}\right) + GM\sqrt{\frac{r_1 - r_0}{r_1 + r_0}}. \tag{4.29}$$

When $r_0 << r_1$, this becomes

$$T \simeq 2M \ln\left(\frac{2r_1}{r_0}\right). \tag{4.30}$$

For more on the topic of relativistic scattering and how it's related to quantum field theories, you can read Chapter 4 of my final year project on Gravitational Time Delay at liamkavanagh.ie/project. There's also many papers in the References worth reading.

5 Black Holes

5.1 Extended Spacetimes

If we consider the Schwarzschild metric, (4.1), null geodesics satisfy

$$\left(\frac{dt}{dr}\right)^2 = \left(\frac{r}{r-2M}\right)^2. \tag{5.1}$$

Here, we have set $G = 1$ for convenience. It can be recovered by the transformation $M \to GM$. The solutions here are

$$t = \pm r_* + c, \quad r_* = r + 2M \ln\left(\tfrac{r}{2M} - 1\right) \tag{5.2}$$

for some constant c. We can move to a transverse coordinate system by letting

$$\begin{aligned} u &= t - r_* \\ v &= t + r_*. \end{aligned} \tag{5.3}$$

Then, the Schwarzschild metric can be written in terms of u, v and the original r as

$$ds^2 = -\tfrac{2M}{r} e^{-r/2M} e^{(v-u)/4M} du dv \tag{5.4}$$

Defining

$$\begin{aligned} U &= -e^{-u/4M} \in (-\infty, 0) \\ V &= e^{v/4M} \in (0, \infty), \end{aligned} \tag{5.5}$$

the metric becomes

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV. \tag{5.6}$$

Because nothing funny happens when U or V go to zero, discontinuities and the like, we can extend this spacetime to the full region $(-\infty, \infty)$. Changing coordinates once again to

$$\begin{aligned} T &= \frac{U+V}{2}, \\ X &= \frac{V-U}{2}, \end{aligned} \tag{5.7}$$

we get

$$ds^2 = \frac{32M^2}{r} e^{-r/2M} (-dT^2 + dX^2). \tag{5.8}$$

This spacetime is shown in Fig. 7. Note that $UV = -(\frac{r}{2M} - 1)e^{-r/2M}$. As $r \to 0$, the metric curvature blows up and we get a singularity. This occurs at $UV = 1$. The surface of $r = 2M$, i.e. $UV = 0$ is the event horizon.

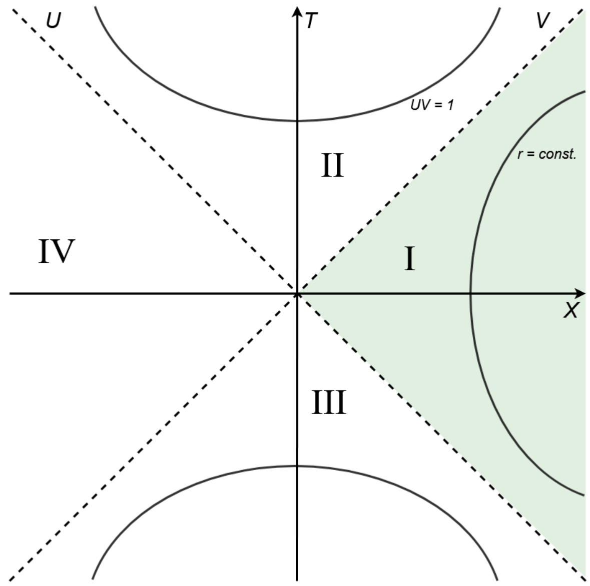


Figure 7: A diagram of the extended spacetime, showing the four different spacetime regions. Null geodesics are lines $T = \pm X + c$. An observer in region II cannot communicate with an observer in region I as it would require propagation of light outside the light cone.

At $t = 0$ along the equatorial plane, the Schwarzschild metric is

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\phi^2. \tag{5.9}$$

In cylindrical coordinates, we can identify $\rho(r) = r$ and $z(r) = \pm\sqrt{8M(r - 2M)}$. This is the metric of a wormhole existing instantaneously at $t = 0$.

5.2 Charged and Rotating Black Holes

The Schwarzschild solution describes an electrically neutral, non-rotating black hole. A charged black hole can be found by solving the Einstein field equations with the electromagnetic stress-energy tensor. The solution is the Reissner-Nordström metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2, \tag{5.10}$$

where $\Delta = 1 - \frac{2GM}{r} + \frac{G}{r^2}(Q^2 + P^2)$. Here, Q and P are the total electric and magnetic charges[§], respectively.

Depending on the parameters, any of three scenarios may occur. The event horizon is specified by the equation $\Delta = 0$, which has solutions

$$r_\pm = M \pm \sqrt{M^2 - (Q^2 + P^2)}. \tag{5.11}$$

If $M^2 < Q^2 + P^2$, then there is no event horizon, but there is a "naked" singularity at $r = 0$.

If $M^2 > Q^2 + P^2$, there is an infinite chain of horizons connecting asymptotic spacetimes (see Fig. 8).

[§]Note that usually $P = 0$ as magnetic monopoles have never been observed.

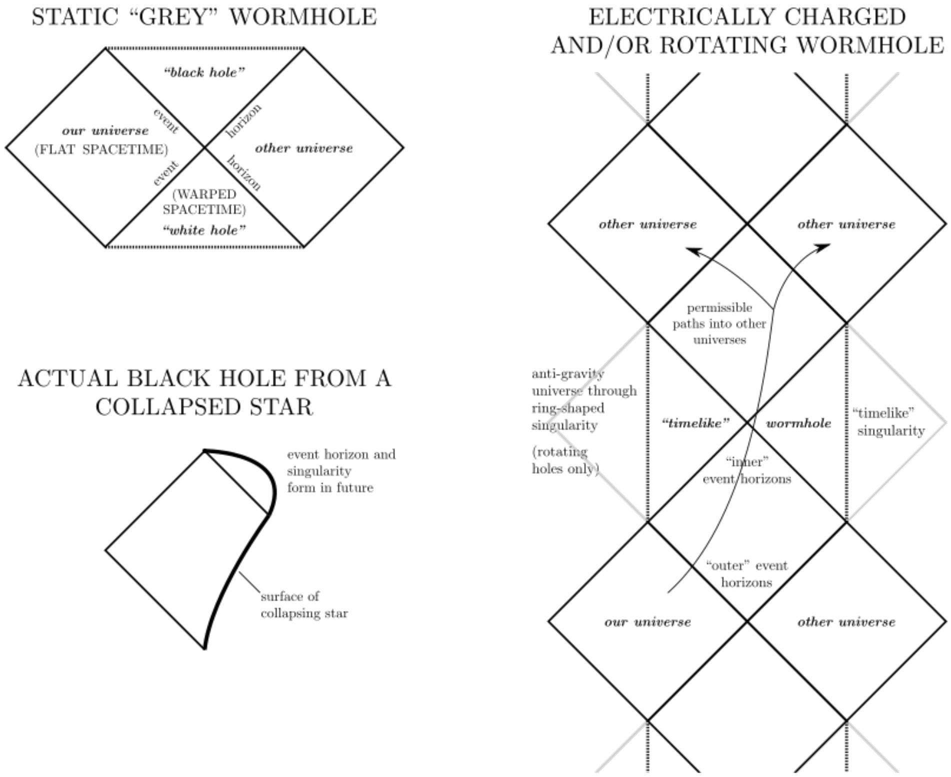


Figure 8: Several Penrose diagrams for black holes. Source: AkanoToE.

A rotating neutrally-charged black hole is described by the (somewhat unwieldy) Kerr metric

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{2Mar\sin^2\theta}{\rho^2} 2dtd\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2. \tag{5.12}$$

Here, we have the parameters $\Delta = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $a = \dfrac{J}{M}$, the ratio of the angular momentum to the mass. As $a \to 0$, we recover the Schwarzschild metric and as $M \to 0$ we recover Minkowski spacetime.

The singularity of a Kerr black hole is a ring, occuring at $\theta = \dfrac{\pi}{2}$ and the horizon is at $\Delta = 0$. There are two horizon surfaces, found at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

There are two Killing vectors generating symmetries, namely ∂_t and ∂_ϕ. In the case of ∂_t, we have

$$|\partial_t|^2 = -\frac{1}{\rho^2}(\Delta - a^2\sin^2\theta). \tag{5.13}$$

We have $|\partial_t|^2 = 0$ on the stationary limit surface defined by $(r-M)^2 = M^2 - a^2\cos^2\theta$. This is the boundary of the ergosphere, a region where there are no stationary observers. In order to remain on a timelike trajectory, particles must rotate with the black hole.

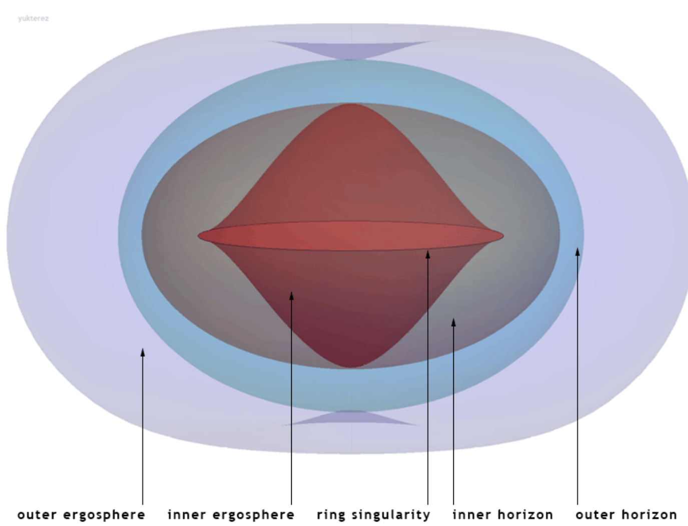


Figure 9: The features of a Kerr black hole. Source: Yukterez (Simon Tyran, Vienna).

6 Cosmology

6.1 The Friedmann–Lemaître–Robertson–Walker Metric

The Friedmann-Lemaître-Robertson-Walker (FLRW)[¶] metric describes an isotropic universe whose size changes over time. It is given by

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{6.1}$$

where $k = 0$ is flat space, $k = +1$ is a closed (spherical) universe, and $k = -1$ is an open (hyperbolic) universe.

The Einstein field equations for a perfect fluid of density ρ and pressure P, with an additional cosmological constant is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}. \tag{6.2}$$

The solutions are the **Friedmann equations**

$$\begin{aligned} \frac{\dot{a}^2 + k}{a^2} &= \frac{8\pi G \rho + \Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}. \end{aligned} \tag{6.3}$$

Introducing the Hubble parameter $H = \frac{\dot{a}}{a}$ and the density parameter $\Omega = \frac{8\pi G\rho}{3H^2}$, we can write the first equation (with $\Lambda = 0$) as

$$\Omega - 1 = \frac{k}{H^2 a^2}. \tag{6.4}$$

By measuring Ω we can determine which kind of universe we live in (open, closed, or flat).

Often an assumption is made that the equation of state is $P = \omega\rho$. This is a good assumption for dealing with a homogeneous dust ($\omega = 0$) or radiation-filled space ($\omega = 1/3$).

[¶] Sometimes called the Friedmann metric or FRW metric, or many other combinations. Naming everyone seemed the least controversial. Also, Lemaître's name has historically been dropped from many concepts he helped create, such as the Hubble-Lemaître Law, so I thought it important to include him.

6.2 The Cosmological Constant

With the stress-energy tensor of a cosmological constant $\Lambda g_{\mu\nu}$, we have a condition on the density and pressure that

$$\rho = -P = \frac{\Lambda}{8\pi G}. \tag{6.5}$$

This gives the cosmological constant an equation of state $P = \omega\rho$ with $\omega = -1$.

The continuity equation $\nabla_\mu T^\mu_\nu = 0$ gives the equation

$$3\frac{\dot{a}}{a}(\rho + P) = -\dot{\rho}. \tag{6.6}$$

Using the equation of state $P = \omega\rho$, this equation has solutions of the order

$$\rho \sim a^{-3(1+\omega)}. \tag{6.7}$$

In the three cases of dust, radiation, and a cosmological constant we get

ω	Source	Density
0	dust	$\rho \sim a^{-3}$
1/3	radiation	$\rho \sim a^{-4}$
-1	Λ	$\rho \sim$ const.

We see that the cosmological term dominates in the expansion.