# MAU34402 - Classical Electrodynamics
# Brief Notes

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## 1 A Covariant Formulation of the Maxwell Equations

### 1.1 Lorentz Transformations

Lorentz transformations[1] are the linear transformations from one spacetime coordinate system, or reference frame, $\mathcal{S}$ (with space and time coordinates $t$, $x$, $y$, $z$) to another reference frame $\mathcal{S}'$, such that the two frames are moving with a constant velocity $v$ relative to one another. Intuitively, the inverse transformation from $\mathcal{S}'$ to $\mathcal{S}$ is the Lorentz transformation with velocity $-v$.

Similar to how reflections, rotations, and translations of the coordinate system in Euclidean space preserve the Euclidean metric

$$d^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2,$$

the Lorentz transformations preserve the *spacetime metric*,

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2.$$

A Lorentz transformation from $\mathcal{S}$ to $\mathcal{S}'$ such that the two sets of axes are parallel (i.e. no rotational change) is called a Lorentz boost. A boost in the $x$ direction by a velocity $v$ is characterised by

$$t' = \gamma(t - \frac{v}{c^2}x), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z,$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the Lorentz factor. This can be written equivalently in terms of a dimensionless velocity $\beta = \dfrac{v}{c}$ such that $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, and the same boost is characterised by

$$ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z.$$

We can see more clearly now that the transformation rules for a boost are in terms of the components of the spacetime metric $s^2$. Instead of using the usual vector notation $\vec{x} = (x, y, z)$, which is really only convenient in euclidean space, we can rewrite these equations in terms of four-vector notation, which includes the time component.

[1]Named after Hendrik Lorentz (1853-1928). Despite being most well known for the Lorentz transformations underpinning Special Relativity, he won the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery of the Zeeman effect.

## 1.2 4-Vector Notation

Four-vector notation is similar to usual vector notation, except there are four components and we have to account for the signs of terms. We can see that the spacetime metric has a minus sign on all of the spatial terms. Relabelling the components of the spacetime metric as $ct = x_0, x = x_1, y = x_2, z = x_3$, we can write the spacetime metric as

$$s^2 = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2.$$

If we now define a four vector as

$$x_\mu = (ct, -\vec{x}) = (x_0, x_1, x_2, x_3),$$

where $\mu$ represents the components $0, 1, 2, 3$, we have what's called a "covariant" vector with the index lowered. We could almost write the spacetime metric as a dot product of two of these covariant four-vectors, we just need to have a minus sign on the last three terms. We can accomplish this using what's called the *metric tensor*, which is really just a fancy name for a $4 \times 4$ matrix, defined as

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The components of this matrix are labelled $g^{\mu\nu}$, where $\mu$ and $\nu$ range from 0 to 3. We see that unless $\mu = \nu$, $g^{\mu\nu} = 0$. What happens if we multiply our four-vector by this metric tensor? We get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}.$$

Which is just our initial four-vector with minus signs exactly where we need them! We call this a "contravariant" four-vector, which is simply

$$x^\mu = (ct, \vec{x}) = (x^0, x^1, x^2, x^3).$$

Now, we can write the spacetime metric as a product not of two four-vectors, but a four-vector and a four-vector multiplied by the metric tensor!

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2.$$

Now, this is rather cumbersome to write out every single time, especially when so many components are zero. Instead, we write this in terms of components of the four-vector and the metric tensor;

$$s^2 = x_\mu g^{\mu\nu} x_\nu = x_\mu x^\mu,$$

where we sum over repeated indices, as we would for a usual dot product of three-vectors:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i.$$

In this case however, the dot product is over four indices, from zero, and the minus signs are included;

$$x_\mu x^\mu = \sum_{\mu=0}^{3} x_\mu \sum_{\nu=0}^{3} g^{\mu\nu} x_\nu = x_0(+x_0) + x_1(-x_1) + x_2(-x_2) + x_3(-x_3) = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2.$$

We can also use this four-vector notation to rewrite our Lorentz transformation rules as

$$x'^0 = \gamma(x^0 - \beta x^1), \quad x'^1 = \gamma(x^1 - \beta x^0)$$

with the other two components remaining unchanged. For a boost in an arbitrary direction, we have

$$x'_0 = \gamma(x_0 - \vec{\beta} \cdot \vec{x}), \quad \vec{x}' = \vec{x} + \left(\frac{\gamma - 1}{\beta^2}\vec{\beta} \cdot \vec{x} - \gamma x_0\right) \cdot \vec{\beta}.$$

Note that Lorentz transformations are now defined as the linear transformations leaving the so-called Minkowski product $x_\mu x^\mu$ invariant. The components of $x'^\mu$ are the transformed components of $x^\mu$ such that

$$x'^\mu = \Lambda^\mu_\nu x^\nu = \Lambda^\mu_0 x^0 + \Lambda^\mu_1 x^1 + \Lambda_2 x^2 + \Lambda_3 x^3,$$

where we require $x'_\mu x'^\mu = x_\mu x^\mu$. Thus, we have

$$x'_\mu x'^\mu = g_{\mu\nu} x'^\nu x'^\mu = g_{\mu\nu}(\Lambda^\nu_\alpha x^\alpha)(\Lambda^\mu_\beta x^\beta) = x^\alpha x^\beta \Lambda^\mu_\beta \Lambda^\nu_\alpha g_{\mu\nu}.$$

Remember that this product must be invariant, so we have to have

$$g_{\alpha\beta} = \Lambda^\mu_\beta g_{\mu\nu} \Lambda^\nu_\alpha.$$

In matrix notation, this would be $g = \Lambda^T g \Lambda$. If we take the determinant, we get

$$\det(g) = \det\left(\Lambda^T g \Lambda\right) = \det\left(\Lambda^T\right) \det(g) \det(\Lambda) \implies 1 = \det(\Lambda)^2.$$

Thus, we must have $\det(\Lambda) = \pm 1$. We can also find by taking the 00 component that we need $|\Lambda^0_0| \geq 1$. The Lorentz transformations form a group represented by the $4 \times 4$ matrices with determinant $\pm 1$ and $|\Lambda^0_0| \geq 1$. We can define four categories of Lorentz transforms based on these two properties. We will discuss the "proper orthochronous" Lorentz group, where the parity of the time and spatial components are left unchanged.

## 1.3 The Field Strength Tensor

In vector notation, the Maxwell Equations are (in Gaussian cgs units)

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0,$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} := \frac{4\pi}{c}\vec{j}.$$

Differentiating the first of these equations, we find

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{c}\frac{\partial \rho}{\partial t},$$

and taking the divergence of the last, we get

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right) = \frac{4\pi}{c}\vec{\nabla} \cdot \vec{j}.$$

Since the divergence of a curl is zero, adding the two equations yields the continuity equation

$$\frac{4\pi}{c}\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j}\right) = 0.$$

We want to obtain a covariant formulation for these equations using four-vector notation. Noting that

$$\vec{\nabla} \cdot \vec{j} = \frac{\partial}{\partial x^k} j^k, \quad k = 1, 2, 3,$$

we can now rewrite the continuity equation using the four-current-density $j^\mu = (c\rho, \vec{j})$ such that

$$\frac{\partial}{\partial x^0}(c\rho) + \frac{\partial}{\partial x^k} j^k = \partial_\mu j^\mu.$$

Thus, the continuity equation is simply

$$\partial_\mu j^\mu = 0.$$

Note that $\partial^\mu = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla}\right)$ and $\partial_\mu = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$. The upper index on $\partial^\mu$ corresponds to a lower index on $\frac{\partial}{\partial x_\mu}$ (and vice versa). This is because for a covariant position $x_\mu$, the derivative will vary inversely, i.e. contravariantly, so the operation $\partial^\mu$ will be contravariant.

If we take a look at the homogenous equations, we see that

$$\vec{\nabla} \cdot \vec{B} = 0 \implies \vec{B} = \vec{\nabla} \times \vec{A},$$

for some vector field $\vec{A}$. This is the property that a divergenceless vector field can be written as the curl of another vector field, since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$. Substituting this into the last Maxwell Equation to be used, we find

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0 \implies \vec{\nabla} \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0.$$

Using yet another property of vector fields, we find that since the curl of this field is zero, we can write it as the gradient of a scalar field

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi.$$

We now have the equations

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi.$$

Substituting these into inhomogenous Maxwell equations, we find

$$\vec{\nabla}^2\phi - \frac{1}{c}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho,$$

$$\vec{\nabla}^2\vec{A} - \frac{1}{c}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = -\frac{4\pi}{c}\vec{j}.$$

Note that If we were to let $\vec{A} \to \vec{A} + \vec{\nabla}\psi$ and $\phi \to \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}$ for some scalar field $\psi$, then

$$\vec{\nabla} \times (\vec{A} + \vec{\nabla}\psi) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}\psi = \vec{\nabla} \times \vec{A}$$

and similarly for $\vec{E}$, since the curl of the gradient of any scalar field is zero. Thus, changing $\vec{A}$ and $\phi$ by these factors does not affect the physical $\vec{E}$ and $\vec{B}$ fields, however it does ensure that $\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} =$, which is called the Lorenz gauge condition[2]. Thus, we have the equations

[2]Note the name "Lorenz", not "Lorentz". Two different people! The Lorenz gauge is one particular choice of Lorentz invariant gauges. It can be done because the Maxwell equations are overdetermined, so picking a gauge fixes the redundant degrees of freedom. Other gauges are used often, one common example being the Coulomb gauge, where $\vec{\nabla} \cdot \vec{A} = 0$, which is useful for far field radiation because gives the usual Poisson equation for $\phi$ from electrostatics. This is offset however by a rather ugly expression for the charges. Regardless of the gauge, the same results will be obtained for the physical system.

$$\vec{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{4\pi}{c}(c\rho),$$

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}.$$

We already have the four-current-density $j^\mu = (c\rho, \vec{j})$, so letting $A^\mu = (\phi, \vec{A})$ be a four-potential, we have the Maxwell equations in the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) A^\mu = \frac{4\pi}{c} j^\mu.$$

This differential operator is known as the "d'Alembert operator", or simply "d'Alembertian" and is the Minkowski space equivalent of the Laplace operator. It is sometimes denoted by a box, making the equation

$$\Box A^\mu = \frac{4\pi}{c} j^\mu.$$

We can write the d'Alembertian in the more useful form

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \frac{c^2}{c^2}\frac{\partial^2}{\partial x^0 \partial x_0} + \frac{\partial^2}{\partial x^k \partial x_k} = \partial_\mu \partial^\mu.$$

Notice that in terms of the four-potential, the Lorenz gauge condition becomes

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \vec{\nabla}\vec{A} = \frac{\partial A^0}{\partial x^0} + \frac{\partial A^k}{\partial x^k} = \partial_\mu A^\mu,$$

so the Lorenz gauge condition is $\partial_\mu A^\mu = 0$.

We can now see that with these definitions, we can compute the components of the $\vec{E}$ and $\vec{B}$ fields easily. For example,

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi \implies E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial\phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \implies B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = -(\partial^2 A^1 - \partial^1 A^2).$$

We define the Field Strength Tensor[3] as

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

This rank two tensor (read: matrix) gives us all of the components of the physical electric and magnetic fields in terms of the derivatives of the potentials. Neat! Notice that the field strength tensor is antiymmetric and traceless:

$$F^{\alpha\beta} = -F^{\beta\alpha}, \quad F^{\alpha\alpha} = 0.$$

We can define the covariant form of the field strength tensor using the metric tensor to pull down the indices

$$F_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta}.$$

Computing the elements of $F_{\alpha\beta}$ shows that it's equivalent to swapping only the electric field components of $F^{\alpha\beta}$. i.e. $F_{\alpha\beta} = F^{\alpha\beta}\big|_{\vec{E}\to-\vec{E}}$.

[3] Also called the electromagnetic tensor, electromagnetic field tensor, Faraday tensor, Maxwell bivector... so many names...

We can reclaim the inhomogenous Maxwell equations as

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta,$$

and the homogenous Maxwell equations as

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$$

We can also define the dual field strength tensor as

$$G^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}.$$

This simplifies the form of the homogenous Maxwell equations to an equivalent expression of the inhomogenous equations

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = \partial_\alpha G^{\alpha\beta} = 0.$$

The dual field strength tensor is equivalent to swapping $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ in $F^{\alpha\beta}$. Hence, it corresponds to "magnetic currents", which are not observed, hence the equation is homogenous.

It can be shown by arduous computation that the field strength tensor transforms under a Lorentz transformation as

$$F'^{\alpha\beta} = \Lambda^\alpha_\mu \Lambda^\beta_\nu F^{\mu\nu}.$$

This is unsurprising, as this is how all tensors transform. Indeed, mathematicians in their typical tautological ways can define tensors as "things that transform like a tensor".

This leads to the form of the transformed electric and magnetic fields

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{1+\gamma}(\vec{\beta} \cdot \vec{E})\vec{\beta},$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{1+\gamma}(\vec{\beta} \cdot \vec{B})\vec{\beta}.$$

In the case of a boost in the $x$ direction, i.e. $\vec{\beta} = (\beta, 0, 0)$, we have

$$E'_1 = E_1, \quad E'_2 = \gamma(E_2 - \beta B_3), \quad E'_3 = \gamma(E_3 + \beta B_2),$$

$$B'_1 = B_1, \quad B'_2 = \gamma(B_2 + \beta E_3), \quad B'_3 = \gamma(B_3 - \beta E_2).$$

## 1.4 Lorentz Invariants

We've just seen how the field strength tensor transforms under a Lorentz transformation. However, there are some constructions which are invariant under these transformations. These are the so-called Lorentz scalars:

$$F_{\mu\nu}F^{\mu\nu}, \quad G_{\mu\nu}G^{\mu\nu}, \quad F_{\mu\nu}G^{\mu\nu}.$$

Computing these (again, arduous computation), we get

$$F_{\mu\nu}F^{\mu\nu} = 2(\vec{E}^2 - \vec{B}^2),$$

$$G_{\mu\nu}G^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2),$$

$$G_{\mu\nu}F^{\mu\nu} = 4\vec{B} \cdot \vec{E}.$$

There are two indepenent quantities here, $\vec{E}^2 - \vec{B}^2$ and $\vec{E} \cdot \vec{B}$. These are the two Lorentz invariants for the physical fields. These have some important physical consequences:

1. If $\vec{E}\cdot\vec{B}=0$, then $\vec{E}'\cdot\vec{B}'=0$ in all other reference frames. i.e. if the electric and magnetic fields are perpendicular in one frame, they will be perpendicular in all frames.
2. If $\vec{E}^2=\vec{B}^2$, then $\vec{E}'^2=\vec{B}'^2$, i.e. the magnitudes of the fields are always equal, or if $|\vec{E}|\lessgtr|\vec{B}|$, then $|\vec{E}'|\lessgtr|\vec{B}'|$.
3. If $\vec{E}\cdot\vec{B}=0$, then there exists a reference frame such that either $\vec{E}'=0$ or $\vec{B}'=0$, depending on $\vec{E}'-\vec{B}'\lessgtr 0$.

## 1.5 Energy and Momentum of an EM Field

The work done by an EM field on a charge current is given by

$$\mathcal{P}=\int_V d^3\vec{x}\vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t).$$

Here's a thought: why doesn't this depend on the magnetic field?

For example, one could consider the trivial case of a moving point charge, $\vec{j}(\vec{x},t)=q\vec{v}(t)\delta^{(3)}(\vec{x}-\vec{x}(t))$. More generally, we can use the Maxwell equations to give

$$\frac{4\pi}{c}\vec{j}=\vec{\nabla}\times\vec{B}-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\implies\vec{j}\cdot\vec{E}=\frac{c}{4\pi}\left[\vec{E}\cdot(\vec{\nabla}\times\vec{B})-\frac{1}{c}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right].$$

*Mathematical Interlude:*

$$\begin{aligned}\vec{\nabla}(\vec{E}\times\vec{B})&=\nabla_k(E_lB_m)\varepsilon_{klm}\\&=\left[(\nabla_kE_l)B_m+E_l(\nabla_kB_m)\right]\varepsilon_{klm}\\&=\vec{B}\cdot(\vec{\nabla}\times\vec{E})-\vec{E}\cdot(\vec{\nabla}\times\vec{B}).\end{aligned}$$

Thus,

$$\vec{j}\cdot\vec{E}=\frac{c}{4\pi}\left[\vec{B}\cdot(\vec{\nabla}\times\vec{E})-\vec{\nabla}(\vec{E}\times\vec{B})-\frac{1}{c}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right],$$

but by Maxwell's equations, $\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$, so we have

$$\begin{aligned}\vec{j}\cdot\vec{E}&=\frac{c}{4\pi}\left[-\frac{1}{c}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{\nabla}(\vec{E}\times\vec{B})-\frac{1}{c}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right]\\&=\frac{c}{4\pi}\left[-\frac{1}{2c}\frac{\partial}{\partial t}\left(\vec{E}^2+\vec{B}^2\right)-\vec{\nabla}(\vec{E}\times\vec{B})\right]\\&=-\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{E}^2+\vec{B}^2\right)-\frac{c}{4\pi}\vec{\nabla}(\vec{E}\times\vec{B}).\end{aligned}$$

The work done by the field is now

$$\mathcal{P}=\int_V d^3\vec{x}\left[\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{E}^2+\vec{B}^2\right)-\frac{c}{4\pi}\vec{\nabla}(\vec{E}\times\vec{B})\right].$$

Using Stokes' theorem, we can write the second term as a surface integral to get

$$\mathcal{P}=-\frac{\partial}{\partial t}\int_V d^3\vec{x}-\frac{1}{8\pi}(\vec{E}^2+\vec{B}^2)-\frac{c}{4\pi}\int_{\partial V}(\vec{E}\times\vec{B})\cdot\hat{n}da.$$

This result is called Poynting's theorem, and is eqiuivalent to conservation of energy. Defining the Poynting vector as

$$\vec{S}:=\frac{c}{4\pi}(\vec{E}\times\vec{B}),$$

and recalling the energy density is

$$\mathcal{U} = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right),$$

we get

$$\mathcal{P} = \int_v d^3\vec{x}\vec{j}\cdot\vec{E} = -\frac{\partial}{\partial t}\int_V d^3\vec{x}\mathcal{U} - \int_{\partial V}\vec{S}\cdot\hat{n}da.$$

In differential form, this equation is

$$\frac{\partial\mathcal{U}}{\partial t} + \vec{\nabla}\vec{S} = -\vec{j}\cdot\vec{E}.$$

In the absence of sources ($\vec{j} = \vec{0}$), the rate of change of energy in a volume $V$ is equal to the energy flux through the surface $\partial V$ of $V$.

We can also look at the force exerted by the EM field on a charge and current density. This is given by the Lorentz force law.

$$\vec{F} = \int_V d^3\vec{x}(\rho(\vec{x})\cdot\vec{E} + \frac{\vec{j}}{c}\times\vec{B}).$$

For a single point charge at $\vec{x}'$, $\rho(\vec{x}) = q\delta^{(3)}(\vec{x}-\vec{x}')$ and the Lorentz force is

$$\vec{F} = q\vec{E} + \frac{\vec{v}}{c}\times\vec{B}.$$

Recalling that $j^\mu = (c\rho, \vec{j})$, if we let $f^\mu = F^{\mu\nu}j_\nu$, we find that

$$f^0 = F^{0k}j_k = \vec{E}\cdot\vec{j},$$

$$f^k = F^{k0}j_0 + F^{kl}j_l = c\rho E_k - \varepsilon_{klm}B_m j_l = \left[\rho\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]_k.$$

This combines both the work and Lorentz force into the field strength tensor.

Using the fact that $j^\mu = \frac{c}{4\pi}\partial_\nu F^{\nu\mu}$, we have

$$f^\mu = F^{\mu\nu}j_\nu = F^\mu_\nu j^\nu = \frac{c}{4\pi}F^\mu_\nu\partial_\rho F^{\rho\nu}.$$

Using the product rule and the previous cyclic result for the field strength tensor, we can write this as

$$f^\mu = \frac{c}{4\pi}\left[\partial_\rho(F^\mu_\nu F^\rho\nu) - \frac{1}{4}\partial^\mu(F^{\nu\rho}F_{\nu\rho})\right] := -c\partial^\nu T^\mu_\nu,$$

where $T^\mu_\nu$ is the stress-energy-momentum tensor, or stress-energy tensor for short.

$$T^\mu_\nu = \frac{1}{4\pi}\left[F^\mu_\rho F^\rho_\nu - \frac{1}{4}\delta^\mu_\nu F^\rho_\sigma F^\sigma_\rho\right].$$

We find that

$$T_{00} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) = \mathcal{U},$$

$$T_{0k} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_k = \frac{1}{c}S_k,$$

$$T^i_j = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(\vec{E}^2 + \vec{B}^2\right)\right] = \sigma_{ij},$$

the last equation being the Maxwell stress tensor for an EM field. I think these equations make the choice of name - “stress-energy-momentum tensor” - quite obvious.

# 2 Solutions to the Covariant Maxwell Equations

## 2.1 Homogenous Solutions

We now have a covariant formulation for the Maxwell equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu,$$

$$\partial_\mu G^{\mu\nu} = 0 \iff \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0,$$

and we can now attempt to find solutions to these equations. Acting on the homogenous equation with $\partial_\mu$, we find

$$\partial_\mu(\partial^\mu F^{\nu\rho}) + \partial^\nu(\partial_\mu F^{\rho\mu}) + \partial^\rho(\partial_\mu F^{\mu\nu}) = 0.$$

Recognising that $\partial_\mu F^{\rho\mu} = -\partial_\mu F^{\mu\rho}$ and using the inhomogenous equation, we get

$$\partial_\mu \partial^\mu F^{\nu\rho} - \frac{4\pi}{c}\partial^\nu j^\rho + \frac{4\pi}{c}\partial^\rho j^\nu = 0.$$

The quantity $\partial_\mu \partial^\mu$ is the d'Alembert box operator, so we have

$$\Box F^{\nu\rho} = \frac{4\pi}{c}(\partial^\nu j^\rho - \partial^\rho j^\nu).$$

Each component of $F^{\nu\rho}$ satisfies this differential equation. e.g.

$$\Box F^{10} = \Box E_1 = \frac{4\pi}{c}\left(\partial^1 j^0 - \partial^0 j^1\right) = \frac{4\pi}{c}\left(\partial^1(c\rho) - \frac{1}{c}\frac{\partial j_1}{\partial t}\right) = -4\pi\left(\nabla_1 \rho + \frac{1}{c^2}\frac{\partial j_1}{\partial t}\right),$$

$$\Box F^{12} = -\Box B_3 = \frac{4\pi}{c}\left(\partial^1 j^2 - \partial^2 j^1\right) = -\frac{4\pi}{c}\left(\nabla_1 j_2 - \nabla_2 j^1\right).$$

These examples imply that

$$\Box \vec{E} = -4\pi\left(\vec{\nabla}\rho + \frac{1}{c^2}\frac{\partial \vec{j}}{\partial t}\right),$$

$$\Box \vec{B} = \frac{4\pi}{c}\vec{\nabla} \times \vec{j}.$$

If we now consider the four-potential $A^\mu = (\phi, \vec{A})$, we have

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu,$$

$$\implies \frac{4\pi}{c} j^\nu = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu).$$

Recalling the Lorenz gauge condition $\partial_\mu A^\mu = 0$, we have

$$\Box A^\nu = \frac{4\pi}{c} j^\nu, \iff \Box\phi = 4\pi\rho, \quad \Box\vec{A} = \frac{4\pi}{c}\vec{j}.$$

These are wave equations of the form

$$\Box\psi(\vec{x},t) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\psi(\vec{x},t) = 4\pi f(\vec{x},t).$$

If we consider the homogenous equation, where $j^\mu = 0 \iff f(\vec{x},t) = 0$, we have

$$\Box\psi(\vec{x},t) = 0$$

which is the wave equation for an electromagnetic field in vacuum. The solutions are the monochromatic plane waves

$$\psi(\vec{x},t) = e^{-i(\vec{k}\cdot\vec{x} \pm \omega t)}$$

corresponding to waves moving in the $\mp\frac{\vec{k}}{k}$ direction.

## 2.2 Green's Functions and the Inhomogenous Wave Equation

Perhaps the most general way of thinking about solving equations such as the inhomogenous wave equation (in terms of the four-vector x)

$$\Box\psi(x) = 4\pi f(x)$$

is to consider a Green's function. Finding a Green's function of a differential operator is basically equivalent to finding an inverse operator. In the case of matrices and vectors, one could imagine having the equation

$$A\vec{v} = \vec{u}$$

where only $A$ and $\vec{u}$ are known. If we can find $A^{-1}$, then since $AA^{-1} = I$, we know

$$A(A^{-1}\vec{u}) = I\vec{u} = \vec{u},$$

which implies that $A^{-1}\vec{u}$ is a solution of the equation $A\vec{v} = \vec{u}$. We can use a similar technique for the case of differential operators, which are linear operators like matrices. If we consider the wave equation, a Green's function $D_0$ for the d'Alembert operator will satisfy

$$\Box D_0(x - x') = \delta^{(4)}(x - x'),$$

which would mean that

$$\psi(x) = \int D_0(x - x')4\pi f(x')d^4x'$$

is as solution, since

$$\Box \int D_0(x - x')4\pi f(x')d^4x' = \int \Box D_0(x - x')4\pi f(x')d^4x' = \int \delta^{(4)}(x - x')4\pi f(x')d^4x' = 4\pi f(x).$$

In general a Green's function will not be unique, and indeed we find two Green's functions for this wave equation

$$D_{\text{adv}}(x) = -\Theta(-x_0) \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\sin\left(|\vec{k}|x_0\right)}{|\vec{k}|},$$

$$D_{\text{ret}}(x) = \Theta(x_0) \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\sin\left(|\vec{k}|x_0\right)}{|\vec{k}|},$$

where $\Theta(x)$ is the Heavyside step function. The naming schemes will become clearer when we discuss the properties of each. We can explicitly evaluate these Green's functions to find

$$D_{\text{ret}}(x - x') = -\frac{1}{4\pi|\vec{x} - \vec{x}'|}\delta((x_0 - x_0') - |\vec{x} - \vec{x}'|),$$

$$D_{\text{adv}}(x) = D_{\text{ret}}(-x) = \frac{1}{4\pi|\vec{x} - \vec{x}'|}\delta((x_0 - x_0') + |\vec{x} - \vec{x}'|).$$

These Green's functions measure the response[4] felt at a point $(x_0, \vec{x})$ from the field generated at the point $(x_0', \vec{x}')$. Note that this fluctuation of the field travels through both space and time. The retarded Green's function measures responses propagating forward in time, i.e. $t - t' > 0$ and the advanced Green's function measures responses propagating backwards in time, i.e. $t - t' < 0$.[5]

[4] The Green's function is in fact exactly a measure of how the system responds to an instantaneous pulse, hence the use of a delta function. It is also called a two-point correlation function, or "propagator" in QFT, since it measures the response to a fluctutation at one point in spacetime at a second point in spacetime, i.e. how a wave "propagates".

[5] The advanced Green's function may seem unphysical, since we usually take initial conditions and want to see how a system evolves forward in time. It is actually used when we know the end state of a system and want to see where the field came from. These are the kinds of calculations used in particle colliders like those at CERN, where the end products and fields of particle collisions is known and one wants to work backwards to find the different interactions that took place. Of course, this treatment must usually done using a Quantum Field Theory, rather than a Classical Field Theory.

Defining a retarded time $t_{\text{ret}} = t - \frac{1}{c}|\vec{x} - \vec{x}'|$, we can write the retarded Green's function as

$$D_{\text{ret}}(\vec{x} - \vec{x}', t - t') = \frac{\delta(t_{\text{ret}} - t')}{4\pi|\vec{x} - \vec{x}'|}.$$

Similarly, an advanced time $t_{\text{adv}} = t + \frac{1}{c}|\vec{x} - \vec{x}'|$ yields

$$D_{\text{adv}}(\vec{x} - \vec{x}', t - t') = \frac{\delta(t_{\text{adv}} - t')}{4\pi|\vec{x} - \vec{x}'|}.$$

# 3 Moving Point Charges

## 3.1 Liénard-Weichert Potential

Now that we have an expression for our Green's functions, we can attempt to find solutions of the inhomogenous Maxwell equations

$$\Box A^\mu(x) = \frac{4\pi}{c} j^\mu(x).$$

We can find solutions if $j^\mu(x)$ is known. One very important case is that of a moving point charge, where

$$\rho(\vec{x}, t) = q\delta^{(3)}(\vec{x} - \vec{r}(t)),$$
$$\vec{j}(\vec{x}, t) = q\vec{v}(t)\delta^{(3)}(\vec{x} - \vec{r}(t)),$$

where $\vec{r}(t)$ is the trajectory of the point charge and $\vec{v}(t)$ is its velocity. Defining a four-velocity $U^\mu = (c, \vec{v}(t))$, we can write these equations as

$$j^\mu(x) = (\rho, \vec{j}) = \frac{q}{\gamma} U^\mu \delta^{(3)}(\vec{x} - \vec{r}(t)).$$

Introducing the four-trajectory $r^\mu(\tau)$ parametrised by the proper time $\tau$ of the point charge such that $r^0(\tau) = ct,\ r^k(\tau) = \vec{r}_k(t)$, we have, with $d\tau = \frac{dt}{\gamma(t)}$,

$$j^\mu(x) = \int \frac{dt}{\gamma} \delta(t - \frac{1}{c} r^0(\tau)) \prod_{k=1}^{3} \delta(x_k - r^k(\tau)) U^\mu(\tau) = qc \int d\tau \delta^{(4)}(x - r(\tau)) U^\mu(\tau).$$

However, we know that the solutions to the wave equation are given by

$$A^\mu(x) = \frac{4\pi}{c} \int D_{\text{ret}}(x - x') j^\mu(x') d^4x',$$

so we have the equation

$$\begin{aligned} A^\mu(x) &= qc\frac{4\pi}{c} \int d^4x' D_{\text{ret}}(x - x') \int d\tau \delta^{(4)}(x' - r(\tau)) U^\mu(\tau) \\ &= qc\frac{4\pi}{c} \int d^4x' \frac{1}{2\pi} \Theta(x_0 - x_0') \delta[(x - x')^2] \int d\tau \delta^{(4)}(x' - r(\tau)) U^\mu(\tau) \\ &= 2q \int d\tau \Theta(x_0 - r_0(\tau)) \delta[(x - r(\tau)^2] U^\mu(\tau). \end{aligned}$$

Noting that we only get contributions to $A^\mu$ when $(x - r(\tau_0))^2 = 0$ in the delta function for some $\tau_0$, we can find the contributing values using the generalised scaling property of the Dirac delta,

$$\int f(x)\delta(g(x))dx = \sum_{g(x_i)=0} \frac{f(x_i)}{|g'(x_i)|}.$$

This yields the equation

$$A^\mu(x) = \frac{qU^\mu(\tau_0)}{U^\rho(\tau_0)(x - r(\tau_0))^\rho}$$

which is the Liénard-Weichert Potential. From this, we can compute the $\vec{E}$ and $\vec{B}$ fields due to the moving point charge.

### 3.2 Generated Electromagnetic Fields

Now that we know the four-potential $A^\mu(x)$ for a moving point charge, called the Liénard-Weichert potential, we can find the electric and magnetic fields from the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Using our expression for $A^\mu(x)$, we can find by using integration by parts and the chain rule that

$$F^{\mu\nu}(x) = 2q \int d\tau \Theta(x_0 - r_0(\tau))\delta[(x - r(\tau))^2] \frac{d}{d\tau} \left[ \frac{(x - r(\tau))^\mu U^\nu - (x - r(\tau))^\nu U^m u}{U_\rho (x - r(\tau))^\rho} \right].$$

Noting as before that the only contribution comes from $(x - r(\tau_0))^2 = 0$, we have

$$F^{\mu\nu}(x) = \frac{q}{U_\rho(\tau_0)(x - r(\tau_0))^r ho} \frac{d}{d\tau} \left[ \frac{(x - r(\tau))^\mu U^\nu(\tau) - (x - r(\tau))^\nu U^\mu}{U_\rho(\tau)(x - r(\tau))^\rho} \right]_{\tau=\tau_0}.$$

For a particle trajectory $r^\mu(\tau)$, $r^0(\tau_0) = ct_{\text{ret}}$. Defining the quantity $R = |x - r(\tau_0)|$, we have

$$(x - r(\tau_0))^\mu = R(1, \hat{n})$$

for a unit vector $\hat{n}$. Also, we have from its definition

$$U^\mu = \gamma(c, \vec{v}) = \gamma c(1, \vec{\beta}) \implies \frac{dU^\mu}{d\tau} = \gamma c(\dot{\gamma}, \dot{\gamma}\vec{\beta} + \gamma\dot{\vec{\beta}}).$$

We thus have

$$U_\rho (x - r(\tau))^\rho = \gamma c(1, \vec{\beta}) \cdot R(1, \hat{n}) = \gamma c R(1 - \vec{\beta} \cdot \hat{n}).$$

Evaluated at $t_{\text{ret}}$, the electric field is given by

$$E_k = \underbrace{\frac{q}{R^2\gamma^2(1 - \vec{\beta} \cdot \hat{n})^3}(\hat{n} - \vec{\beta})_k}_{\text{velocity field component}} + \underbrace{\frac{q}{Rc(1 - \vec{\beta} \cdot \hat{n})^3} \left( \hat{n} \cdot \dot{\vec{\beta}}(\hat{n} - \vec{\beta})_k - (1 - \vec{\beta} \cdot \hat{n})\dot{\beta}_k \right)}_{\text{acceleration field component}}.$$

This can also be written as

$$E_k = \frac{q}{R^2\gamma^2(1 - \vec{\beta} \cdot \hat{n})^3}(\hat{n} - \vec{\beta})_k + \frac{q}{Rc(1 - \vec{\beta} \cdot \hat{n})^3} \left[ \hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right]_k.$$

### 3.3 The Relativistic Larmor Formula

This expression for the electric field leads to an equation for the power radiated by the moving charge,

$$P(t) = \frac{2}{3}\frac{q^2}{c} \left[ \gamma^6 \left( \dot{\vec{\beta}}^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right) \right]_{t_{\text{ret}}},$$

which is the relativistic Larmor formula for the power radiated, including relativistic effects. It can also be written in terms of the four-momentum $p^\mu = (\frac{\mathcal{E}}{c}, \vec{p})$ as

$$P(t) = -\frac{2}{3}\frac{q^2}{m^2c^3} \left[ \frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} \right]_{t_{\text{ret}}}.$$

In the non-relativistic limit, $|\vec{\beta}| << 1$, the equation becomes

$$P(t) \simeq \frac{2}{3}\frac{q^2}{c}\dot{\vec{\beta}}^2.$$

There are some special cases of the Larmor formula, one being that of linear motion where the velocity and acceleration are parallel, i.e. $\vec{\beta} \parallel \dot{\vec{\beta}}$. In this case,

$$P(t) = -\frac{2}{3}\frac{q}{c}\,\gamma^6\dot{\vec{\beta}}\Big|_{t_{\mathrm{ret}}} = \frac{3}{2}\frac{q^2}{m^2c^3}\left(\frac{d\mathcal{E}}{dx}\right)^2\Bigg|_{t_{\mathrm{ret}}}.$$

This gives the power radiated/lost per distance travelled due to rectilinear motion. We see that accelerating a charge has an added cost to overcome energy loss due to the particle's motion.

Another special case is circular motion, where the velocity and acceleration are perpendicular, i.e. $\vec{\beta} \perp \dot{\vec{\beta}} \implies \vec{\beta}\cdot\dot{\vec{\beta}} = 0$. In this case, $v = |\vec{v}| = \omega r$ is constant. If the velocity is

$$\vec{v}(t) = (r\cos(\omega t), r\sin(\omega t), 0),$$

we have $\dot{\vec{v}}(t) = -\omega^2\vec{v}(t)$ and thus

$$P(t) = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\omega^4r^2.$$

The loss of energy per period is

$$\Delta\mathcal{E} = \frac{2\pi}{\omega}P(t) = \frac{4\pi}{3}q^2\beta^3\frac{\gamma^4}{r}.$$

Classically, an electron orbiting in an atom would be unstable. It was postulated that the electron would radiate no energy in order to keep a stable orbit. The need for this assumption was removed by treating the energies of electrons quantum mechanically, with fixed angular momentum.

## 3.4 Angular Distribution of Radiation

The energy radiated in an area $R^2 d\Omega$ is given by $d\mathcal{E} = [\vec{S}\cdot\hat{n}R^2]_{t_{\mathrm{ret}}} d\Omega dt_{\mathrm{ret}}$.

$$\implies \frac{d\mathcal{E}}{d\Omega} = \int_{t_1}^{t_2} [\vec{S}\cdot\hat{n}R^2(1-\vec{\beta}\cdot\hat{n})]_{t_{\mathrm{ret}}} dt = \int_{t_1}^{t_2}\frac{dP}{d\Omega}dt.$$

where we have used $t_{\mathrm{ret}} = (1-\vec{\beta}\cdot\hat{n})t$. Supposing that $\vec{\beta}$ and $\dot{\vec{\beta}}$ vary slowly over this time interval, we have

$$\frac{d\mathcal{E}}{d\Omega} = \Delta t\frac{dP}{dt} = \Delta t\frac{q^2}{4\pi c}\left[\frac{(\hat{n}\times((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}))^2}{(1-\vec{\beta}\cdot\hat{n})^5}\right].$$

For linear motion, $\vec{\beta} \parallel \dot{\vec{\beta}}$, the radiation in direction $\hat{n}$ making an angle $\theta$ with the direction of motion is

$$\frac{dP}{d\Omega} = \frac{q^2|\dot{\vec{\beta}}|^2}{4\pi c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \underset{\beta<<1}{\simeq} \frac{q^2|\dot{\vec{\beta}}|^2}{4\pi c}\sin^2\theta.$$

## 3.5 The Case of a General Source Charge

We now consider the radiation from a general source $j^\mu = (c\rho, \vec{j})$. The solution to the wave equation for this source is

$$A^\mu_{\mathrm{ret}}(x) = \frac{4\pi}{c}\int d^4x' D_{\mathrm{ret}}(x-x')j^\mu(x'),$$

so we have

$$\rho_{\mathrm{ret}}(\vec{x},t) = \int d^3\vec{x}'\frac{1}{|\vec{x}-\vec{x}'|}\rho(t-\tfrac{1}{c}|\vec{x}-\vec{x}'|,\vec{x}'),$$

$$\vec{A}_{\mathrm{ret}}(\vec{x},t) = \frac{1}{c}\int d^3\vec{x}'\frac{1}{|\vec{x}-\vec{x}'|}\vec{j}(t-\tfrac{1}{c}|\vec{x}-\vec{x}'|,\vec{x}').$$

In the far field, for $r := |\vec{x}| >> |\vec{x}'|$, we have

$$\vec{A}_{\text{ret}}(t,\vec{x}) \simeq \frac{1}{rc}\int d^3\vec{x}'\vec{j}(t-\tfrac{r}{c},\vec{x}').$$

Using the continuity equation $\dot{\rho} + \vec{\nabla}\cdot\vec{j} = 0$, we have

$$\vec{A}_{\text{ret}}(t,\vec{x}) = \frac{1}{rc}\frac{d}{dt}\int d^3\vec{x}'\rho(t-\tfrac{r}{c},\vec{x}')\vec{x}'.$$

However, this integral is simply the electric dipole moment for the charge density $\rho$. Thus,

$$\vec{A}_{\text{ret}}(t,\vec{x}) \simeq \frac{1}{rc}\dot{\vec{d}}(t-\tfrac{r}{c}).$$

From this we can find the electric and magnetic fields to be

$$\vec{B} \simeq -\frac{1}{rc^2}\hat{n}\times\dot{\vec{d}}(t-\tfrac{r}{c}),$$

$$\vec{E} \simeq -c\hat{n}\times\vec{B} = \frac{1}{rc^2}\hat{n}\times(\hat{n}\times\ddot{\vec{d}}(t-\tfrac{r}{c}).$$

The radiated power is found from the Poynting vector and is

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} \simeq \frac{1}{4\pi r^2c^2}\left|\hat{n}\times\ddot{\vec{d}}\right|^2\hat{n}.$$

Since $\vec{S} \parallel \hat{n}$, the power is radiated radially, but not uniformly. Assuming the dipole oscillates in the $z$-direction, then

$$\vec{S} = \frac{1}{4\pi r^2c^2}\left|\ddot{\vec{d}}\right|^2\sin^2\theta\hat{n}.$$

The total power radiated is

$$P = \int d^2\vec{r}\cdot\vec{S} = \frac{2}{3}\frac{\left|\ddot{\vec{d}}\right|^2}{c^2}.$$

# 4 Electromagnetism in Linear Media

## 4.1 The Displacement and Magnetising Fields

Up to now, we've been considering the Maxwell equations in a vacuum,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}.$$

Defining the displacement field $\vec{D} = \varepsilon_0\vec{E}+\vec{P}$ where $\vec{P} = \varepsilon_0\chi_e\vec{E}$ is the polarisation of the linear medium, and defining the magnetising field $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ where $\vec{M} = \frac{1}{\mu_0}\frac{\chi_m}{\chi_m+1}\vec{B}$ is the magnetisation of the linear medium, we have

$$\vec{\nabla}\cdot\vec{D} = \rho_{\text{free}}, \quad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{H} = \vec{j}_{\text{free}} + \frac{\partial\vec{D}}{\partial t}.$$

Note that $\rho_{\text{free}}$ and $\vec{j}_{\text{free}}$ are the free current density and current not bound by the medium, $\chi_e$ is the electric susceptibility, and $\chi_m$ is the magnetic susceptibility. We can also write these in terms of $\varepsilon = (1+\chi_e)\varepsilon_0$, the permittivity of the medium, and $\mu = (1+\chi_m)\mu_0$, the permeability of the medium to get

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{H} = \frac{1}{\mu}\vec{B}.$$

## 4.2 Waves in Dielectric Media

In insulators and dielectric media, $\rho_{\text{free}} = 0$ and $\vec{j}_{\text{free}} = \vec{0}$. Thus, we have

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0,$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}, \quad \vec{\nabla} \times \vec{B} - \mu\varepsilon \frac{\partial \vec{E}}{\partial t} = \vec{0}.$$

Differentiating the third equation and substituting for the fourth, we find expressions for the wave equation for $\vec{E}$ and $\vec{B}$;

$$\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \vec{\nabla}^2 \vec{E} = \vec{0}, \quad \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} - \vec{\nabla}^2 \vec{B} = \vec{0},$$

where the wave velocity is $v = \frac{1}{\sqrt{\mu\varepsilon}}$. We can find the solutions to these wave equations with the ansatz

$$\vec{E}(\vec{x}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)},$$

however we must remember that the *physical* electric field is the real part of this expression. Inserting the plane wave ansatz, we find

$$-\frac{1}{v^2}(-i\omega)^2 - (i\vec{k})^2 = 0 \implies \vec{k}^2 = \frac{\omega^2}{v^2}.$$

We can find from the Maxwell equations that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies i\vec{k} \times \vec{E} = i\omega\vec{B} \implies \vec{k} \times \vec{E} = \omega\vec{B},$$

$$\vec{\nabla} \cdot \vec{E} \implies \vec{k} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \implies \vec{k} \cdot \vec{B} = 0.$$

Thus, the wavevector $\vec{k}$ is orthogonal to both $\vec{E}$ and $\vec{B}$, which are also orthogonal to each other. Note also that since $\vec{B} = \vec{k} \times \vec{E}$, the magnetic field is entirely determined by the electric field.

## 4.3 Polarisation of Plane Waves

If we choose $\vec{k}$ to be along the $z$-axis, then we have $\vec{E}_0 = (E_{0,x}, E_{0,y}, 0)$, where

$$E_{0,x} = |E_{0,x}|e^{i\varphi}, \quad E_{0,y} = |E_{0,y}|e^{i(\varphi+\delta)}.$$

We now distinguish some cases:

1. Linear polarisation: $\delta = 0$ or $\delta = \pm\pi$. We have

   $$\text{Re}[\vec{E}(\vec{x}, t)] = (|E_{0,x}|\hat{x} \pm |E_{0,y}|\hat{y}) \cos(kz - \omega t + \phi).$$

2. Circular polarisation: $\delta = \pm\frac{\pi}{2}$. For $|E_{0,x}| = |E_{0,y}| = E_0$, we have

   $$\text{Re}[\vec{E}(\vec{x}, t)] = E_0(\hat{x}\cos(kz - \omega t + \varphi) \mp \hat{y}\sin(kz - \omega t + \varphi).$$

   A phase offset of $\delta = \frac{\pi}{2}$ corresponds to right-circular polarisation, and $\delta = -\frac{\pi}{2}$ corresponds to left-circular poalrisation.

3. Elliptic polarisation: $\delta = \pm\frac{\pi}{2}$, with $|E_{0,x}| \neq |E_{0,y}|$.

4. Tilted elliptic polarisation: $\delta$ takes an arbitrary value and $|E_{0,x}| \neq |E_{0,y}|$. The ellipse is tilted away from being perpendicular to the $z$-axis.

## 4.4 Energy Transport in Plane Waves

In SI units, the Poynting vector in a medium is given by

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu}\vec{E} \times \vec{B}.$$

The energy density is given by

$$\mathcal{U} = \frac{1}{2}\left(\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{B}\right) \underset{\text{linear media}}{=} \frac{1}{2}\left(\frac{1}{\mu}\vec{B}^2 + \varepsilon \vec{E}^2\right).$$

Both of these equations assume that we have already taken the real parts of $\vec{E}$ and $\vec{B}$, since in general

$$\mathrm{Re}[\vec{E}] \times \mathrm{Re}[\vec{B}] \neq \mathrm{Re}[\vec{E} \times \vec{B}].$$

If the time dependence of a quantity is harmonic, i.e. $\propto e^{i\omega t}$, which is the case for the electric and magnetic fields, then we can use time averaging over one period of oscillation, $T$. We define a time average

$$\langle f(t) \rangle = \frac{1}{T}\int_t^{t+T} f(t')dt'.$$

With this definition, we have

$$\mathrm{Re}[\vec{E}] \cdot \mathrm{Re}[\vec{B}] = \frac{1}{2}\mathrm{Re}[\vec{E} \cdot \vec{B}^*].$$

Thus, we have

$$\left\langle \vec{S} \right\rangle = \frac{1}{2}\mathrm{Re}[\vec{E} \times \vec{H}^*] \underset{\text{linear media}}{=} \frac{1}{2\mu}\mathrm{Re}[\vec{E} \times \vec{B}^*],$$

$$\langle \mathcal{U} \rangle = \frac{1}{4}\mathrm{Re}[\vec{H} \cdot \vec{B}^* + \vec{E} \cdot \vec{D}^*] \underset{\text{linear media}}{=} \frac{1}{4}\left(\frac{1}{\mu}|\vec{B}|^2 + \varepsilon |\vec{E}|^2\right).$$

Note also that

$$\left\langle \vec{S} \right\rangle = v\, \mathcal{U}\hat{k}.$$

## 4.5 Reflection and Transmission

At the interface of a two linear dielectric media, e.g. air and glass, air and water, etc. with wave velocities $v_1$ and $v_2$, we have an incident, reflected, and transmitted wave given by

$$\vec{E}_{I,R,T}(\vec{x}, t) = \vec{E}_{0,I,R,T}e^{i(\vec{k}_{I,R,T} \cdot \vec{x} - \omega_{I,R,T}t)}, \quad \vec{B}_{I,R,T}(\vec{x}, t) = \frac{\vec{k}_{I,R,T}}{v_{1,1,2}} \times \vec{E}_{I,R,T}(\vec{x}, t).$$

The integral form of the Maxwell equations and the conditions that the waves must be continuous at the boundary give the results

$$\omega_I = \omega_R = \omega_T := \omega,$$

$$\vec{k}_I,\ \vec{k}_R,\ \vec{k}_T \quad \text{all in the same plane (First Law of Optics)},$$

$$k_I = k_R \implies \sin\theta_R = \sin\theta_I \quad \text{(Law of Reflection)},$$

$$k_I \sin\theta_I = k_T \sin\theta_T \implies \frac{n_1}{n_2} = \frac{\sin\theta_T}{\sin\theta_I} \quad \text{Snell's Law of Refraction},$$

where $n = \dfrac{c}{v}$ is the refractive index of the medium.